Linear Algebra and Differential Equations Chapter Summaries

The following connects many ideas from different portions of the course.

Let A be an $n \times n$ matrix. The following are equivalent:

(a) The linear system of equations $A\boldsymbol{x} = \boldsymbol{0}$ has solutions in addition to $\boldsymbol{x} = \boldsymbol{0}$
(b) A is not invertible
(c) $\det(A) = 0$
(d) the reduced row echelon form of A has at least one row of zeros
(e) the reduced row echelon form of A has fewer than n pivots
(f) $\text{rank}(A) < n$
(g) the null space of A contains non-zero vectors
(h) the columns of A are linearly dependent
(i) 0 is an eigenvalue of A

It is also useful to notice that the product of an $m \times n$ matrix $A = (a_{ij})$ with a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^t$ is simply a notation for writing the columns of A as a linear combination.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}$$
$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$$

Chapter 5

A *vector space* is a set of vectors which is closed under addition (the sum of any two vectors in the space is in the space) and scalar multiplication (the product of a scalar and a vector is in the space) and for which the following eight properties hold: (A1) addition of vectors is commutative; (A2) addition of vectors is associative; (A3) there is an additive identity (the zero vector); (A4) each vector has an additive inverse; (M1) multiplication by scalars is associate; (M2) there is a scalar mulitplicative identity (1); (D1) distributive law for scalars; (D2) distributive law for vectors.

Examples of vector spaces include sets of arrays of numbers of the same dimension, sets of functions, sets of matrices, and so on.

A vector *subspace* is a subset of a vector space that is a vector space in its own right when it inherits the operations of addition and scalar multiplication from the original vector space. To show that a subset is a subspace, it is sufficient to show that it is closed under addition and scalr multiplication.

The *null space* of an $m \times n$ matrix A is the subset of $\mathbf{R}^n$ that satisfies the homogeneous system of equations $A\boldsymbol{x} = \boldsymbol{0}$. The null space of A is a subspace of $\mathbf{R}^n$. Other books use the name *kernel* for the null space.

The vector $\boldsymbol{v}$ is a *linear combination* of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ if $\boldsymbol{v} = \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_k\boldsymbol{v}_k$ for some scalars $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$.

The *span* of a set of vectors is the set of all linear combinations of the vectors in the set. The span of a set of vectors in a vector space is a vector subspace.

If A is an $m \times n$ matrix with rank r, then the null space of A is a span of $n - r$ vectors.

A vector $\boldsymbol{v}$ is in the span of vectors $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k\}$ if and only if there is a solution $\boldsymbol{x}$ to the system of equations $A\boldsymbol{x} = \boldsymbol{v}$ where $A = (\boldsymbol{w}_1|\cdots|\boldsymbol{w}_k)$. This will be true if and only if the reduced row echelon form of the matrix $(\boldsymbol{w}_1|\cdots|\boldsymbol{w}_k|\boldsymbol{v})$ does not have a pivot in the last column.

A set of vectors is *linearly dependent* if one of the vectors can be written as a linear combination of the others. The set of vectors is *linearly independent* if no vector in the set is a linear combination of the others. The set of vectors $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k\}$ is linearly independent if and only if the only solution to the equation

$$x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + \cdots + x_k\boldsymbol{w}_k = 0$$

is $x_1 = \cdots = x_k = 0$. If $\boldsymbol{w}_i \in \mathbf{R}^m$, the condition above is equivalent to the statement that the null space of the matrix $A = (\boldsymbol{w}_1|\cdots|\boldsymbol{w}_k)$ is the zero vector in $\mathbf{R}^k$.

A vector space has finite dimension if it is the span of a finite number of vectors. The size of the smallest possible spanning set is called the *dimension* of the vector space.

A set of vectors is a *basis* for a vector space if it spans the vector space and no set with fewer vectors spans the vector space. A set will be a basis for a vector space if and only if it spans the vector space and its elements are linearly independent.

The *nullity* of an $m \times n$ matrix A is the dimension of the null space of A. If the reduced row echelon form of A has r pivots, then the rank of A is r and the nullity of A is $n - r$. Consequently, the nullity of A plus the rank of A is n.

Chapter 6

The system of autonomous ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$ will have a unique solution when the functions f and g are differentiable in a neighborhood of (x_0, y_0) and when the partial derivatives of f and g with respect to x and y individually are continuous in a neighborhood of the point (x_0, y_0).

If the linear system of n equations $\dot{X} = CX$ with C an $n \times n$ matrix has n solutions $X_1(t), \ldots, X_n(t)$ such that the initial conditions $\boldsymbol{v}_j = X_j(0)$ are linearly independent, then the general solution is of the form

$$X(t) = \alpha_1 X_1(t) + \cdots + \alpha_n X_n(t)$$

and a particular solution with initial condition $X(0) = X_0$ is of the form above where

$$X_0 = \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_n\boldsymbol{v}_n$$

See Professor Taylor's handouts for additional chapter summary information.

Chapter 8

The *determinant* of a square $n \times n$ matrix is a real number that satisfies these three properties: (a) the determinant of a lower triangular matrix is the product of the diagonal elements; (b) the determinant of the transpose of the transpose of a matrix is the determinant of the matrix (c) the determinant of a product of two square matrices is the product of the determinants. The determinant function of the 2×2 matrix $ad - bc$ is the only function that satisfies the properties above. For general $n \times n$ matrices, the properties above define a determinant function that exists and is unique.

The elementary row operations of swapping two rows, multiplying a row by a non-zero constant, and replacing a row by itself plus a constant time another row are all equivalent to invertible matrix multiplications. The determinant of a swap matrix is -1. The determinant of a matrix which multiplies a row by c is c. The determinant of a matrix which replaces a row by the row plus a constant times another row is 1. To find the determinant of an $n \times n$ matrix A, one strategy is to convert A to a triangular matrix E by elementary row operations. Then $E = R_s R_{s-1} \cdots R_1 A$ and $\det A = \det E / (\det R_1 \cdots \det R_s)$.

The matrix A_{ij} is the $(n-1) \times (n-1)$ matrix for which the $i$the row and jth column of A have been deleted. The matrix $(-1)^{i+j} A_{ij}$ is called a *cofactor matrix* of A The determinant of an $n \times n$ matrix A may be written as a linear combination of the determinants of the cofactor matrices along a row or column of A. To expand by cofactors along the ith row,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

To expand by cofactors along the jth column,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

A good method to calculate determiants by hand is to use elementary row operations to add 0s to a row or column and then to expand by cofactors along that row or column.

The complex number λ is an *eigenvalue* of the square $n \times n$ matrix A if $A\boldsymbol{v} = \lambda \boldsymbol{v}$ for an *eigenvector* $\boldsymbol{v}$.

The characteristic polyniomial of a square $n \times n$ matrix A is $p_A(\lambda) = \det(A - \lambda I)$. This is an n degree polynomial of the form

$$\begin{aligned} p_A(\lambda) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \\ &= (-1)^n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_1 \lambda + b_0 \end{aligned}$$

The constant $b_{n-1} = (-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = (-1)^{n-1}\mathrm{tr}(A)$. The constant $b_0 = \lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$. The matrix A will have exactly n eigenvalues (counting multiplicity) that are either real or come in complex conjugate pairs. A is invertible if and only if A does not have 0 as an eigenvalue. If matrices A and B are similar, then the matrices have the same characteristic polynomials and, hence, the same eigenvalues and the same trace. The converse is not true — there are matrices with the same characteristic polynomial that are not similar. The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ each have characteristic polynomial $(1-\lambda)^2$ for example, but are not similar. The identity matrix is similar only to itself.