Lecture 4 – Sampling Theory

CS559 2007

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Notes not for reference, not for display

Outline

Last time

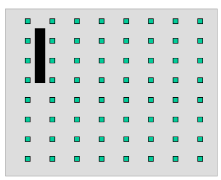
- Intensity/Quantization
- Gamma Correction
- Discrete sensing and displays
- Point samples
- Sampling, Reconstruction, Resampling, Aliasing

This time

- What can go wrong with sampling
- Sampling Theory Intuitions
- Some sampling theory math
- Sampling theory without the math

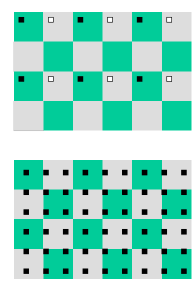
Bad sampling is bad

- Miss small things between samples



Get really weird results

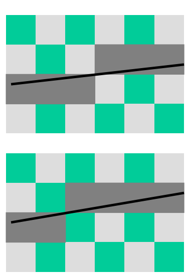
- Sample a checkerboard
  - Look at a sampled picture
- Too few samples
  - Get all black
  - Get all white
  - Get weird patterns
    - Aliasing
    - Moire'
  - Arbitrary algorithm decision gives very different answers!
- Imagine resampling



Demonstration ratios: 4/6 (here) = 2/3

Ugly

- Imagine line drawing
- Jaggies
- Crawlies
  - Small change causes jump
  - Smooth motion becomes jumpy

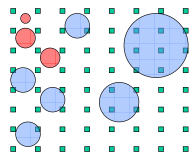


Dealing with discretization

- Sampling
  - Understand what information we are throwing away
- Reconstruction
  - Recreate as well as possible from the samples
- Re-Sampling
  - Transform the image
- Signal Processing / Image Processing
- Consider the 1D case first since its easier

Intuition

- Too few samples = BAD
- Sampling rate depends on the thing you're sampling
- Need to sample close enough to get smallest object
- Need to limit small objects to be big enough that they aren't missed

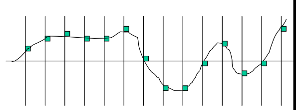


A different intuition

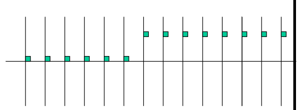
- Not really point sampling
  - Mesurements average over a finite range
  - Displays make finite dots
- Need to model these
  - Sampling filters, reconstruction filters
  - Averages over regions -> Convolution (generalized)
- Need to be realistic about what they mean
  - Can't see everything (too small, …)
- Sampling theory gives a nice mathematics for this!

Point sampling in 1D

- Only record samples
- Don't know what happens in between samples

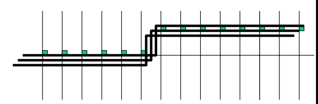


- Given the samples, don't know what really happened!

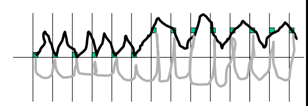


Reconstruction from Sampling

- Can't localize events
  - Bigger problems than that
- No idea! Signal could be anything



- Without additional information, we're guessing as to what the signal is
- But what additional info?



Sampling Intuitions

- Reconstruct the "smoothest" signal that makes sense from samples
- If signal is "smooth enough", sampling will give something we can reconstruct
- If signal is not "smooth", sampling will give something that will reconstruct to something else
  - Aliasing
- But how do we define "smooth"

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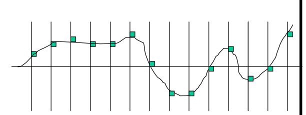

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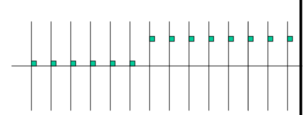

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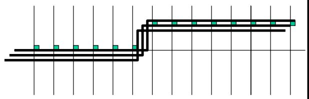

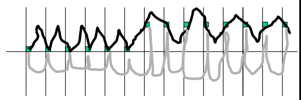

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Signal processing

- Need better "language" for talking about signals
- Idea: represent signals in a different way
- Up till now: time domain (graph against time)
  - Good for asking "what does signal do at time X"
- New idea: frequency domain
  - Good for talking about how smooth signals are
- Different view of the same thing

Frequency Domain

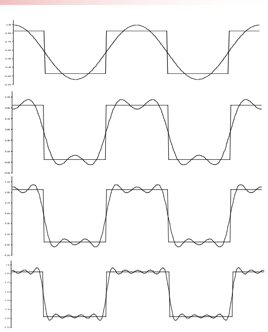
- Fourier Theorem:
  - Any periodic signal can be represented as a sum of sine and cosine waves with harmonic frequencies
  - If one function has frequency *f*, then its harmonics are function with frequency *nf* for integer *n*
  - Extensions to non-periodic signals later
  - Also works in any dimension (e.g. 2 for images, 3, …)
- Example: box

Example: Box (Square Wave)

- 1 cosine – bad
- More cosines, better approx

$$f(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

$$S_{boxes}(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos(2k-1)\omega x}{2k-1}$$
$$= \frac{1}{2} + \frac{2}{\pi}\left(\cos \omega x - \frac{1}{3}\cos 3\omega x + \frac{1}{5}\cos 5\omega x - \cdots\right)$$



Intuitions

- Low frequencies are smooth
  - High frequencies change fast, are not smooth
- If a signal can be made of only low frequencies, it is smooth
- If a signal has sharp changes, it will require high frequencies to represent

Fourier Transform

- $F(\omega)$ is the Fourier Transform of f(t)
  - A different representation of the same signal
  - Express as sums of sins and cosines
- To get f(t) back you use the Inverse Fourier Transform
- You don't need to know how to compute them

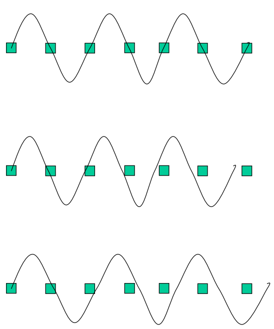
$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Qualitative Properties

- The spectrum of a functions tells us the relative amounts of high and low frequencies
  - **Sharp edges give high frequencies**
  - **Smooth variations give low frequencies**
- A function is *bandlimited* if its spectrum has no frequencies above a maximum limit
  - sin, cos are band limited
  - Box, Gaussian, etc are not
- To band-limit a signal we *low-pass filter* it

Sampling Theorem (intuition)

- High frequencies get lost
  - Can only sample band limited signals
- Sampling rate must be 2 times higher than signal
- Signal must be half frequency of sample rate
  - Otherwise, signal can "turn around" between samples
- Nyquist rate
  - 2x highest frequency in signal



Sampling Theorem

- If your signal is bandlimited
- And you know what the band limit is
- And you sample at (at least) twice that frequency
  - Above the Nyquist rate
- Then – you can reconstruct your signal EXACTLY!

- Caveat
  - Ideal reconstruction requires perfect band limiting in both sampling and reconstruction

Sampling theory in practice

- When you're sampling- PREFILTER
  - Make sure no high frequencies
  - Need to remove them BEFORE sampling
  - Otherwise, aliasing
  - Filtering effectively means blurring
- When you're reconstructing – FILTER
  - View as a spike chain (remove HF)
  - Filtering effectively means interpolating

Theory vs. Practice

Theory	Practice
• Properly sampled original	• Who knows about source?
• Know bandlimit	• Assume that its OK?
• Band-limit signals	• Ideal LPF not practical
• Use Ideal Filters	• Use approximations
• Ideal Reconstructions	• Tradeoffs for "ideal" – Might look blurry – Might want aliasing (sharpness) – Care about efficiency

What is a filter anyway?

- Frequency filters
  - Add remove different frequencies
- Multiplication in frequency means CONVOLUTION in time/space
- Continuous and Discrete Convolutions