CS 787
Fall 2010
Homework #2

Due October 25, 2010

Rules for Homework. See Homework #1.

1. The Ford-Fulkerson method for finding a maximum flow will always halt if the capacities are integers. By expressing capacities as integral multiples of some common denominator, we can see that the same is true when the capacities are rational numbers. This problem deals with networks whose capacities are arbitrary real numbers.
   a) Consider the recurrence $x_{n+2} = x_n - x_{n+1}$. Find $r \in (0, 1)$ such that $x_n = r^n$ satisfies the recurrence.
   b) Consider the flow network, with vertices $s, t, a, b, c, a', b', c'$, and nonzero capacities indicated by the following table:

	s	a	b	c	a'	b'	c'	t
s		∞	∞	∞				
a			∞	∞	x_0			
b		∞		∞		x_1		
c		∞	∞				x_2	
a'						∞	∞	∞
b'					∞		∞	∞
c'					∞	∞		∞
t								

   As usual s is the source and t is the sink. Construct an infinite sequence of flow augmentations such that after the k-th augmentation, the values of the residual capacities $c_f(a, a')$, $c_f(b, b')$, and $c_f(c, c')$ are 0, x_k, and x_{k+1} in some order.
   c) Modify the network so that your answer to b) is still a valid sequence of Ford-Fulkerson augmentations, but the flows produced by this sequence do not converge to a maximum flow.

2. Consider a situation where balls are thrown randomly into bins numbered $1, 2, \ldots, n$. The time at which a fraction α of the bins have at least one ball is a random variable, which we will call $T_{n,\alpha}$.
   a) Find accurate approximations for $E[T_{n,\alpha}]$. As a consequence, you should obtain the well known result that $E[T_{n,1}] \sim n \ln n$. [Hint: analyze the waiting time for a configuration with i bins occupied to evolve to one with $i + 1$ bins occupied, and use linearity of expectation.]
   b) Now, suppose that the probability of landing in bin i is p_i. (Part a) dealt with the case $p_i = 1/n$.) Design and analyze a dynamic programming algorithm to determine the expected time (number of balls thrown) at which all bins have at least one ball.

3. The following questions deal with baseball elimination.

   a) According to folklore, situations in which ties are allowed (e.g. pre-1996 college footbal) can be handled by making each game worth 2 points, and awarding 2,1, or 0 points for a win, tie, or loss. Prove this.

   b) The Elias rule removes a team if it could not possibly win as many games as the current leader. Consider leagues with $n = 2, 3, 4, \ldots$ teams. For which n will the Elias rule correctly determine if a team is eliminated?

   c) (*) In his paper, Wayne notes that the set R of team nodes on the source side of a minimum cut can be used to give a quick proof of elimination. The choice of R is reminiscent of the proof of the König-Egervary theorem. Is there a connection? In particular, can you give an interpretation to the set R' of pair nodes that are on the sink side of the cut?

4. As presented in class, the Hungarian algorithm for the job assignment problem consists of two stages: i) Decrease all entries in a given row or column, and repeat until each row and each column has at least one 0; ii) While there is no matching of 0's, do pivot steps.

   a) Explain why the algorithm will still work if we skip stage i) and go right to ii).

   b) Suppose the array entries c_{ij} are real numbers, chosen independently and uniformly from the interval $[0, 1]$. Show that there is a positive constant C such that the expected size of the matching found after stage i) is at least Cn. Try to get the best C you can.

   From b) we can conclude that i), which is easy, is also worth doing.