CS 787
Fall 2010
Homework #2

Due in class Monday, December 6, 2010

Rules for Homework. See Homework #1.

1. When $M = (E, \mathcal{I})$ is a matroid, the collection $\mathcal{I}$ of independent sets must satisfy three axioms, the first of which states that $\mathcal{I}$ is nonempty.
   a) Why can't I be empty? Nonemptiness presumably prevents some embarrassment, but which one exactly?
   b) Let $|E| = n$. Prove that if the elements of E are assigned distinct weights $w_1, w_2, \ldots w_n$, then the maximum-weight independent subset is unique. (Note: The choices for the greedy algorithm are forced. What you will have to prove is that there isn't some other sneaky way to produce a different max-weight independent set.)
   c) Now suppose that each w_i is assigned a random integer x with $-n \leq x \leq n$. Show that with probability $\geq 1/2$, the max-weight independent subset is unique.
   d) (*) Explore the implications of replacing n by functions $f(n)$ with various growth rates. For example, if $f(n) = n^2$, you should be able to exploit the birthday paradox. How small can you make $f(n)$ and still get a result like c) (possibly for a restricted class of matroids)?

2. Chemists have discovered that carbon atoms can form large spherical molecules called *buckyballs*. If one thinks of the atoms as vertices and the bonds as edges, each buckyball can be represented as a 3-regular planar graph. The faces are either hexagons or pentagons. (See below.)

   a) For any planar graph, the number of vertices, edges, and faces satisfies Euler's formula $V - E + F = 2$. Use this to show that the number of pentagons is divisible by 12.
   b) Are any of the five buckyballs above bipartite?
   c) Do buckyballs have perfect matchings, and if so, how many? (Chemistry suggests that they do.) Answer this question for at least one of the buckyballs above.

3. These questions deal with the Cipolla algorithm for finding a square root of a modulo an odd prime p.

   a) As originally conceived, it was a deterministic algorithm: try $x = 1, 2, 3, 4, \ldots$, until $x^2 - 4a$ is a non-square in $\mathbf{Z}_p$. Show that $x \leq p/4 + O(1)$ suffices.

   b) Let $E_1, E_2, \ldots$ be events in some probability space (not necessarily independent). Show that if $\sum_k Pr[E_k] < \infty$, then the probability is 0 that an infinite number of E_k occur. (Hint: apply the union bound to $\Pr[\exists m > n$ such that E_m occurs $]$.)

   c) Let $a = 2$, and let p be the odd primes for which 2 is a quadratic residue mod p. (Number theorists have proved that these are the primes ± 1 mod 8, so $p = 7, 17, 23, 31, \ldots$.) Let $x(p)$ be the least x that works in the algorithm of a). Use the result of b) to make a guess about growth rate of $x(p)$, and then compare to some empirical data. (You will probably want to use mathematical software like Maple or Mathematica to do your computations.)

   Note: It is an open problem to derandomize Cipolla's algorithm, even if powerful number-theoretic assumptions like ERH are allowed.

4. This question is a little open-ended, and deals with a problem called *proportional representation* in the literature. A nation of total population P is divided into m districts, comprising $p_1, \ldots, p_m$ people. We wish to select n representatives, so that the i-th district is represented by

$$q_i = n\frac{p_i}{P}$$

   of them. (Currently in the U.S. Congress we have $m = 50$ and $n = 435$.) However, since human beings can't be split into fractions (King Solomon notwithstanding), we must somehow choose integral quantities $\hat{q}_i$, $i = 1, \ldots, m$.

   A number of deterministic solutions have been proposed in the literature. For example, the following straightforward greedy procedure is called Hamilton's method: Round each q_i down to an integer and give the i-th district that many representatives, to start. Then sort the fractions

$$f_i = q_i - \lfloor q_i \rfloor, i = 1, \ldots, m,$$

   and assign the remaining positions one by one, in this order.

   a) Using your choice of formal model, investigate the chance that all the q_i will be integral. You should be able to show, at least, that this event is unlikely.

   b) In the U.S., redistricting is done every 10 years, which is ample time for almost any algorithm. However, we can consider a situation in which the p_i's change rapidly. Design and analyze a method that would work well under these circumstances.