1. Biological background.

There are whole genome expression profiles for $n = 128$ tissue samples, divided into four pathologic groups: putatively normal samples, early stage lesions (cervical intraepithelial neoplasia [CIN] 1 and 2), later stage lesions (CIN 3) and frank cancer. Roughly an equal number of tissue samples are in each group. Each tissue sample was measured by an Affymetrix whole genome microarray, which aims to measure the expression of essentially all genes in the genome (it contains about 54,000 probe sets).

The SUCCEED members have already carried out several analyses of these data, including analyses that aim to identify genes showing various patterns of differential expression among the four pathological groupings. However, an alternative analysis can potentially be useful in helping us understand these preliminary findings.

Motivation for the alternative analysis is given by the following biological and technical facts. Each expression profile is measured from a collection of around 1000 cells - so it represents a mixture of theoretical profiles. The stages of cancer progression of cervical tissue are characterized in part by changes in the proportion of cells of particular types. E.g., normal tissue is organized in layers with more well-differentiated cells at the surface and with less differentiated, but more actively dividing cells further inside the tissue. Neoplastic lesions shift the balance of types, at least partly by having relatively more of the less differentiated types and having fewer of the well-differentiated types. Note that the different types will have different gene-expression profiles.

2. Notation and basic definitions.

See the written page R(1).

3. The model.

It is:

$$\begin{aligned} P(\mathbf{X} \in \mathbf{A}) &= E(P(\mathbf{X} \in \mathbf{A}|\mathbf{\Theta})) \\ &= E\Big(\prod_{i=1}^{n} P(X_i \in A_i|\mathbf{\Theta})\Big) \\ &= E\Big(\prod_{i=1}^{n} P\big(\sum_{t=1}^{T} p_{s_i,t} C_{i,t} \in A_i|\mathbf{\Theta}\big)\Big) \end{aligned}$$

3. Mathematical model where we model each tissue as being a mixture of types of cells.

Step 1. For now, we will just consider one fixed gene g. We have a bunch of tissue samples $x_1, \ldots, x_n$ (of that gene) which we think of as realizations of random variables $X_1, \ldots, X_n$. Each sample is from a stage $s_1, \ldots, s_n$ between 1 and 4.

Step 2. Each tissue sample is a mixture of cells whose expression levels we cannot observe directly. We assume that (i) there are T types of cells ($T > 0$ but not too large), (ii) all tissue samples at a particular stage s have the same proportion of cells of type t, (iii) the expression level of a cell depends only on its type. We will use $p_{s,t}$ to denote the proportion of cells of type t at stage s. We will use the random variable $C_{i,t}$ to represent the average expression level of cells of type t in sample i. (From assumption (iii), we have that, for a fixed type t, all the variables $\{C_{i,t}\}_{i=1}^n$ are identical.) Thus:

$$X_i = \sum_{t=1}^{T} p_{s_i,t} C_{i,t}$$

We assume that each type-dependent average cell expression level $C_{i,t}$ follows a Gamma(a, θ_t) distribution, where a is the shape and θ_t is the scale parameter. In other words, $C_{i,t}$ has density

$$f_{\Gamma(a,\theta_t)}(c) = \theta_t^a c^{a-1} \exp(-c\theta_t)/\Gamma(a)$$

Step 3. Next, we want to find a density for the distribution of each X_i. In the previous step, we described how each tissue's expression level is the average of its cells' expression levels, so that $X_i \in A$ if and only if $\sum_{t=1}^{T} p_{s_i,t} C_{i,t} \in A$. We only know the density of each $C_{i,t}$ individually. So we integrate over all possible tuples $\mathbf{c} = (c_1, \ldots, c_T)$ of cell expression levels such that the weighted average $\sum_{t=1}^{T} p_{s_i,t} c_t \in A$. In other words:

$$\begin{aligned}
P(X_i \in A) &= \int_{\mathbb{R}^T} \prod_{t=1}^{T} \left(f_{\Gamma(a,\theta_t)}(c_t) \right) 1_{\{\sum_{t=1}^{T} p_{s_i,t} c_t \in A\} \cap \{c_t \geq 0\}} d\mathbf{c} \\
&= \int_A \int_{\mathbb{R}^T} \prod_{t=1}^{T} \left(f_{\Gamma(a,\theta_t)}(c_t) \right) 1_{\{\sum_{t=1}^{T} p_{s_i,t} c_t = x\} \cap \{c_t \geq 0\}} d\mathbf{c}\, dx \\
&= \int_A f_{\mathrm{one}(i)}(x) dx
\end{aligned}$$

(the exchange of integrals is justified because everything is nonnegative) where the density

$$f_{\mathrm{one}(i)}(x) = \int_{\mathbb{R}^T} \prod_{t=1}^{T} \left(f_{\Gamma(a,\theta_t)}(c_t) \right) 1_{\{\sum_{t=1}^{T} p_{s_i,t} c_t = x\} \cap \{c_t \geq 0\}} d\mathbf{c}$$

Step 4. We assume that tissues' expression levels are independent, so that

$$\begin{aligned}
P(X_1 \in A_1, \ldots, X_n \in A_n) &= \prod_{i=1}^{n} P(X_i \in A_i) \\
&= \prod_{i=1}^{n} \int_{A_i} f_{\mathrm{one}(i)}(x)\, dx \\
&= \int_A \prod_{i=1}^{n} f_{\mathrm{one}(i)}(x_i)\, d\mathbf{x}
\end{aligned}$$

Step 5. We treat each type's scale parameter θ_t as a random variable Θ_t, which follows a prior distribution $\Gamma(a_{0,t}, \nu_t)$. So in Step 4 we actually found

$$P(X_1 \in A_1, \ldots, X_n \in A_n | \Theta_1 = \theta_1, \ldots, \Theta_T = \theta_T)$$

i.e. (notation)

$$P(\mathbf{X} \in \mathbf{A} | \boldsymbol{\Theta} = \boldsymbol{\theta})$$

in Step 3 we found

$$P(X_i \in A_i | \boldsymbol{\Theta} = \boldsymbol{\theta}) \text{ and } f_{\text{one}(i)}(x|\boldsymbol{\theta})$$

and in Step 2 we found

$$f_{\Gamma(a,\Theta_t)}(c)$$

Now we want to again find ("for real") the quantity $P(\mathbf{X} \in \mathbf{A})$:

$$\begin{aligned}
P(\mathbf{X} \in \mathbf{A}) &= E\, P(\mathbf{X} \in \mathbf{A} | \boldsymbol{\Theta}) \\
&= \int_{\mathbb{R}^T} P(\mathbf{X} \in \mathbf{A} | \boldsymbol{\Theta} = \boldsymbol{\theta}) \prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)\, d\boldsymbol{\theta} \\
&= \int_{\mathbb{R}^T} \left[\int_A \prod_{i=1}^{n} f_{\text{one}(i)}(x_i|\boldsymbol{\theta})\, d\mathbf{x} \right] \left[\prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t) \right] d\boldsymbol{\theta} \\
&= \int_A \int_{\mathbb{R}^T} \left[\prod_{i=1}^{n} f_{\text{one}(i)}(x_i|\boldsymbol{\theta}) \right] \left[\prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t) \right] d\boldsymbol{\theta}\, d\mathbf{x} \\
&= \int_A f(\mathbf{x})\, d\mathbf{x}
\end{aligned}$$

(the exchange of integrals is justified because everything is nonnegative) where the "marginal density"

$$\begin{aligned}
f(\mathbf{x}) &= \int_{\mathbb{R}^T} \left[\prod_{i=1}^{n} f_{\text{one}(i)}(x_i|\boldsymbol{\theta}) \right] \left[\prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t) \right] d\boldsymbol{\theta} \\
&= \int_{\mathbb{R}^T} f(\mathbf{x}, \boldsymbol{\theta})\, d\boldsymbol{\theta}
\end{aligned}$$

and the "complete density"

$$f(\mathbf{x}, \boldsymbol{\theta}) = \left[\prod_{i=1}^{n} f_{\text{one}(i)}(x_i|\boldsymbol{\theta}) \right] \left[\prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t) \right]$$

Step 6. We exchange some integrals in order to facilitate an EM algorithm. (Note that the complete density of Step 5 is not easy to evaluate, due to the difficulty of evaluating $f_{\text{one}(i)}$; see "Appendix on simplifying $f_{\text{one}(i)}(x)$" below.)

To do this, observe that in general

$$\prod_{i=1}^{n} \int g_i(x)\,dx = \prod_{i=1}^{n} \int g_i(x_i)\,dx_i$$
$$= \Big(\int g_1(x_1)\,dx_1\Big)\cdots\Big(\int g_n(x_n)\,dx_n\Big)$$
$$= \int \cdots \int g_1(x_1)\cdots g_n(x_n)\,dx_1\cdots dx_n$$

Also recall that we define

$$f_{\mathrm{one}(i)}(x_i|\boldsymbol{\theta};a,\mathbf{c},\mathbf{p}) = \int_{\mathbb{R}^T} \Big(\prod_{t=1}^{T} f_{\Gamma(a,\theta_t)}(c_t)\Big)\Big(1_{\{\sum_{t=1}^{T} p_{s_i,t}c_t = x\}\cap\{c_t\geq 0\}}\Big)\,d\mathbf{c}$$
$$= \int_{\mathbb{R}^T} \mathrm{integrand}(\mathbf{c},x_i,a,\boldsymbol{\theta},\mathbf{p})\,d\mathbf{c}$$

where

$$\mathrm{integrand}(\mathbf{c},x_i,a,\boldsymbol{\theta},\mathbf{p}) = \Big(\prod_{t=1}^{T} f_{\Gamma(a,\theta_t)}(c_t)\Big)\Big(1_{\{\sum_{t=1}^{T} p_{s_i,t}c_t = x\}\cap\{c_t\geq 0\}}\Big)$$

Applying the general observation,

$$\prod_{i=1}^{n} f_{\mathrm{one}(i)}(x_i|\boldsymbol{\theta};a,\mathbf{c},\mathbf{p})$$
$$= \prod_{i=1}^{n} \int_{\mathbb{R}^T} \mathrm{integrand}(\mathbf{c},x_i,a,\boldsymbol{\theta},\mathbf{p})\,d\mathbf{c}$$
$$= \prod_{i=1}^{n} \int_{\mathbb{R}^T} \mathrm{integrand}(\mathbf{c}^{(i)},x_i,a,\boldsymbol{\theta},\mathbf{p})\,d\mathbf{c}^{(i)}$$
$$= \int_{\mathbb{R}^T} \cdots \int_{\mathbb{R}^T} \prod_{i=1}^{n} \mathrm{integrand}(\mathbf{c}^{(i)},x_i,a,\boldsymbol{\theta},\mathbf{p})\,d\mathbf{c}^{(1)}\cdots d\mathbf{c}^{(n)}$$
$$= \int_{\mathbb{R}^{n\times T}} \prod_{i=1}^{n} \mathrm{integrand}(\mathbf{c}^{(i)},x_i,a,\boldsymbol{\theta},\mathbf{p})\,d\text{च}$$

where $\text{च} = (\mathbf{c}^{(1)},\ldots,\mathbf{c}^{(n)}) = (c_{i,t} : i = 1,\ldots,n,\, t = 1,\ldots,T)$.

Plugging this identity into the formula for $f(\mathbf{x})$ from Step 5, we get

$$
\begin{aligned}
&f(\mathbf{x})\\
&= \int_{\mathbb{R}^T} \big[\prod_{i=1}^{n} f_{\mathrm{one}(i)}(x_i|\boldsymbol{\theta};a,\mathbf{c},\mathbf{p})\big]\big[\prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)\big]\,d\boldsymbol{\theta}\\
&= \int_{\mathbb{R}^T} \big[\int_{\mathbb{R}^{n\times T}} \prod_{i=1}^{n} \mathrm{integrand}(\mathbf{c}^{(i)},x_i,a,\boldsymbol{\theta},\mathbf{p})\,d\text{च}\big]\big[\prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)\big]\,d\boldsymbol{\theta}\\
&= \int_{\mathbb{R}^{n\times T}}\int_{\mathbb{R}^T} \big[\prod_{i=1}^{n} \mathrm{integrand}(\mathbf{c}^{(i)},x_i,a,\boldsymbol{\theta},\mathbf{p})\big]\big[\prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)\big]\,d\boldsymbol{\theta}\,d\text{च}\\
&= \int_{\mathbb{R}^{n\times T}}\int_{\mathbb{R}^T} \big[\prod_{i=1}^{n}\big(\prod_{t=1}^{T} f_{\Gamma(a,\theta_t)}(c_{i,t})\big)\big(1_{\{\sum_{t=1}^{T} p_{s_i,t}c_{i,t}=x_i\}\cap\{c_{i,t}\geq 0\}}\big)\big]\\
&\qquad \big[\prod_{t=1}^{T} f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)\big]\,d\boldsymbol{\theta}\,d\text{च}\\
&= \int_{\mathbb{R}^{n\times T}}\int_{\mathbb{R}^T} \prod_{i=1}^{n}\big(1_{\{\sum_{t=1}^{T} p_{s_i,t}c_{i,t}=x_i\}\cap\{c_{i,t}\geq 0\}}\big)\big(\prod_{t=1}^{T} f_{\Gamma(a,\theta_t)}(c_{i,t})\sqrt[n]{f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)}\big)\,d\boldsymbol{\theta}\,d\text{च}\\
&= \int_{\mathbb{R}^{n\times T}} \prod_{i=1}^{n}\big(1_{\{\sum_{t=1}^{T} p_{s_i,t}c_{i,t}=x_i\}\cap\{c_{i,t}\geq 0\}}\big)\int_{\mathbb{R}^T}\big(\prod_{t=1}^{T} f_{\Gamma(a,\theta_t)}(c_{i,t})\sqrt[n]{f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)}\big)\,d\boldsymbol{\theta}\,d\text{च}\\
&= \int_{\mathbb{R}^{n\times T}} \prod_{i=1}^{n}\big(1_{\{\sum_{t=1}^{T} p_{s_i,t}c_{i,t}=x_i\}\cap\{c_{i,t}\geq 0\}}\big) f_{\mathrm{post}}(\mathbf{c}^{(i)};a,\mathbf{a}_0,\boldsymbol{\nu})\,d\text{च}
\end{aligned}
$$

where

$$
f_{\mathrm{post}}(\mathbf{c}^{(i)};a,\mathbf{a}_0,\boldsymbol{\nu}) = \int_{\mathbb{R}^T}\big(\prod_{t=1}^{T} f_{\Gamma(a,\theta_t)}(c_{i,t})\sqrt[n]{f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)}\big)\,d\boldsymbol{\theta}
$$

and $\mathbf{a}_0 = (a_{0,t})_{t=1}^{T}$, $\boldsymbol{\nu} = (\nu_t)_{t=1}^{T}$.

Step 7. We try to simplify f_{post}. Note that

$$
\begin{aligned}
&f_{\mathrm{post}}(\mathbf{c}^{(i)};a,\mathbf{a}_0,\boldsymbol{\nu})\\
&= \int_{\mathbb{R}^T}\prod_{t=1}^{T} f_{\Gamma(a,\theta_t)}(c_{i,t})\sqrt[n]{f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)}\,d\boldsymbol{\theta}\\
&= \int_{\mathbb{R}^T}\prod_{t=1}^{T}\big(\theta_t^{a}c_{i,t}^{a-1}e^{-c_{i,t}\theta_t}/\Gamma(a)\big)\big(\sqrt[n]{\nu_t^{a_{0,t}}\theta_t^{a_{0,t}-1}e^{-\theta_t\nu_t}/\Gamma(a_{0,t})}\big)\,d\boldsymbol{\theta}\\
&= \prod_{t=1}^{T}\int_0^{\infty}\big(\theta_t^{a}c_{i,t}^{a-1}e^{-c_{i,t}\theta_t}/\Gamma(a)\big)\big(\sqrt[n]{\nu_t^{a_{0,t}}\theta_t^{a_{0,t}-1}e^{-\theta_t\nu_t}/\Gamma(a_{0,t})}\big)\,d\theta_t\\
&= \prod_{t=1}^{T}\frac{c_{i,t}^{a-1}\nu_t^{a_{0,t}/n}}{\Gamma(a)\Gamma(a_{0,t})^{1/n}}\int_0^{\infty}\theta_t^{a+(a_{0,t}-1)/n}e^{-c_{i,t}\theta_t-\theta_t\nu_t/n}\,d\theta_t
\end{aligned}
$$

Note that in general, for positive ξ, υ,

$$\int_0^\infty \theta^\xi e^{-\upsilon\theta}\, d\theta = \Gamma(\xi+1)\upsilon^{-\xi-1}$$

So

$$\begin{aligned}
&f_{\text{post}}(\mathbf{c}^{(i)}; a, \mathbf{a}_0, \boldsymbol{\nu}) \\
&= \prod_{t=1}^{T} \frac{c_{i,t}^{a-1} \nu_t^{a_{0,t}/n}}{\Gamma(a)\Gamma(a_{0,t})^{1/n}} \Big(\Gamma((a+(a_{0,t}-1)/n)+1)(c_{i,t}+\nu_t/n)^{-(a+(a_{0,t}-1)/n)-1} \Big) \\
&= \prod_{t=1}^{T} \frac{\Gamma(1+a+(a_{0,t}-1)/n)}{\Gamma(a)\Gamma(a_{0,t})^{1/n}} \frac{c_{i,t}^{a-1} \nu_t^{a_{0,t}/n}}{(c_{i,t}+\nu_t/n)^{1+a+(a_{0,t}-1)/n}}
\end{aligned}$$

CHANGES TO f_{post} HAVEN'T BEEN PROPOGATED PAST HERE.

Step 8. We express the overall density $f(\mathbf{x})$ as the expected value of a density of the distribution of $\mathbf{X}$ conditioned on the cell-type profiles च. Indeed (from Step 6),

$$\begin{aligned}
f(\mathbf{x}) &= \int_{\mathbb{R}^{n\times T}} \prod_{i=1}^{n} \big(1_{\{\sum_{t=1}^T p_{s_i,t} c_{i,t} = x_i\} \cap \{c_{i,t} \geq 0\}} \big) f_{\text{post}}(\mathbf{c}^{(i)}; a, \mathbf{a}_0, \boldsymbol{\nu})\, d\text{च} \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}^T} \big(1_{\{\sum_{t=1}^T p_{s_i,t} c_{i,t} = x_i\} \cap \{c_{i,t} \geq 0\}} \big) f_{\text{post}}(\mathbf{c}^{(i)}; a, \mathbf{a}_0, \boldsymbol{\nu})\, d\mathbf{c}^{(i)} \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}^T} \big(1_{\{\sum_{t=1}^T c_{i,t} = 1\} \cap \{c_{i,t} \geq 0\}} \big) f_{\text{post}}((c_{i,t} x / p_{s_i,t})_{t=1}^T; a, \mathbf{a}_0, \boldsymbol{\nu})\, d\mathbf{c}^{(i)} \\
&= \int_{\mathbb{R}^{n\times T}} \prod_{i=1}^{n} \big(1_{\{\sum_{t=1}^T c_{i,t} = 1\} \cap \{c_{i,t} \geq 0\}} \big) f_{\text{post}}((c_{i,t} x / p_{s_i,t})_{t=1}^T; a, \mathbf{a}_0, \boldsymbol{\nu})\, d\text{च}
\end{aligned}$$

Note that

$$\Delta_T = \{\mathbf{c} \in \mathbb{R}^T : \sum_{t=1}^{T} c_t = 1 \text{ and } c_t \geq 0\}$$

is a standard $(T-1)$-simplex in $\mathbb{R}^T$, so its volume is $\frac{1}{T!}$. See "Appendix on volumes of simplices" below for details. Thus

$$\mathbf{c} \mapsto \frac{1}{T!} 1_{\Delta_T}(\mathbf{c})$$

is a density for a probability measure on $\mathbb{R}^T$, specifically the uniform distribution $\text{unif}(\Delta_T)$, and

$$\begin{aligned}
f_{\text{prior}} : \text{च} \mapsto & \frac{1}{(T!)^n} \prod_{i=1}^{n} 1_{\Delta_T}(\mathbf{c}^{(i)}) \\
&= \frac{1}{(T!)^n} \prod_{i=1}^{n} 1_{\{\sum_{t=1}^T c_{i,t} = 1\} \cap \{c_{i,t} \geq 0\}}
\end{aligned}$$

is a density for the probability measure on $\mathbb{R}^{n\times T}$ which is uniform on $\cap_{i=1}^{n}\{\text{च} \in \mathbb{R}^{n\times T} : \sum_{t=1}^{T} c_{i,t} = 1 \text{ and } c_{i,t} \geq 0\}$.

The point is that we can rewrite $f(x)$ as follows:

$$\begin{aligned}
f(\mathbf{x}) &= \int_{\mathbb{R}^{n\times T}} \prod_{i=1}^{n} \left(1_{\{\sum_{t=1}^{T} c_{i,t}=1\}\cap\{c_{i,t}\geq 0\}}\right) f_{\text{post}}((c_{i,t}x/p_{s_i,t})_{t=1}^{T}; a, \mathbf{a}_0, \boldsymbol{\nu})\, d\text{च} \\
&= \int_{\mathbb{R}^{n\times T}} \left[(T!)^n \prod_{i=1}^{n} f_{\text{post}}((c_{i,t}x/p_{s_i,t})_{t=1}^{T}; a, \mathbf{a}_0, \boldsymbol{\nu})\right] \left[\frac{1}{(T!)^n} \prod_{i=1}^{n} \left(1_{\{\sum_{t=1}^{T} c_{i,t}=1\}\cap\{c_{i,t}\geq 0\}}\right)\right] d\text{च} \\
&= \int_{\mathbb{R}^{n\times T}} f_{\text{cond}}(\mathbf{x}|\text{च}; a, \mathbf{a}_0, \nu, \mathbf{p}) f_{\text{prior}}(\text{च})\, d\text{च}
\end{aligned}$$

where

$$f_{\text{cond}}(\mathbf{x}|\text{च}; a, \mathbf{a}_0, \nu, \mathbf{p}) = (T!)^n \prod_{i=1}^{n} f_{\text{post}}((c_{i,t}x/p_{s_i,t})_{t=1}^{T}; a, \mathbf{a}_0, \boldsymbol{\nu})$$

and f_{prior} is as defined above. In other words,

$$f(\mathbf{x}) = E[f_{\text{cond}}(\mathbf{x}|\text{छ}; a, \mathbf{a}_0, \nu, \mathbf{p})]$$

where छ is a random variable (taking values in $\mathbb{R}^{n\times T}$) with density f_{prior}.

We also define the joint density

$$f(\mathbf{x}, \text{च}; a, \mathbf{a}_0, \nu, \mathbf{p}) = f_{\text{cond}}(\mathbf{x}|\text{च}; a, \mathbf{a}_0, \nu, \mathbf{p}) f_{\text{prior}}(\text{च})$$

Doublecheck.

We doublecheck that

$$\int_{\mathbb{R}^{n\times T}} f(\mathbf{x},\text{च};a,\mathbf{a}_0,\boldsymbol{\nu},\mathbf{p})\,d\text{च} = \int_{\mathbb{R}^T} f(\mathbf{x},\boldsymbol{\theta};a,\mathbf{a}_0,\boldsymbol{\nu},\mathbf{p})\,d\boldsymbol{\theta}$$

Recall (Step 5)

$$\begin{aligned} f(\mathbf{x},\boldsymbol{\theta};a,\mathbf{a}_0,\boldsymbol{\nu},\mathbf{p}) &= \Big[\prod_{i=1}^n f_{\mathrm{one}(i)}(x_i|\boldsymbol{\theta})\Big]\Big[\prod_{t=1}^T f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)\Big] \\ &= \Big[\prod_{i=1}^n \big(\int_{\mathbb{R}^T}\prod_{t=1}^T \big(f_{\Gamma(a,\theta_t)}(c_t)\big)1_{\{\sum_{t=1}^T p_{s_i,t}c_t=x_i\}\cap\{c_t\geq 0\}}d\mathbf{c}\big)\Big]\Big[\prod_{t=1}^T f_{\Gamma(a_{0,t},\nu_t)}(\theta_t)\Big] \end{aligned}$$

while

$$\begin{aligned} f(\mathbf{x},\text{च};a,\mathbf{a}_0,\boldsymbol{\nu},\mathbf{p}) &= f_{\mathrm{cond}}(\mathbf{x}|\text{च};a,\mathbf{a}_0,\boldsymbol{\nu},\mathbf{p})f_{\mathrm{prior}}(\text{च}) \\ &= \Big[(T!)^n\prod_{i=1}^n f_{\mathrm{post}}((c_{i,t}x_i/p_{s_i,t})_{t=1}^T;a,\mathbf{a}_0,\boldsymbol{\nu})\Big]\Big[\frac{1}{(T!)^n}\prod_{i=1}^n 1_{\{\sum_{t=1}^T c_{i,t}=1\}\cap\{c_{i,t}\geq 0\}}\Big] \\ &= \Big[(T!)^n\prod_{i=1}^n\big(\prod_{t=1}^T\frac{\Gamma(a+a_{0,t})}{\Gamma(a)\Gamma(a_{0,t})}\frac{(c_{i,t}x_i/p_{s_i,t})^{a-1}\nu_t^{a_{0,t}}}{((c_{i,t}x_i/p_{s_i,t})+\nu_t)^{a+a_{0,t}}}\big)\Big]\Big[\frac{1}{(T!)^n}\prod_{i=1}^n 1_{\{\sum_{t=1}^T c_{i,t}=1\}\cap\{c_{i,t}\geq 0\}}\Big] \end{aligned}$$

Say $T=2$. Note that $p_{s_i,1}c_1+p_{s_i,2}c_2=x_i$ (i) if $c_1=x_i/p_{s_i,1}$ and $c_2=0$, and (ii) if, for c_1 between 0 and $x_i/p_{s_i,1}$, we have $p_{s_i,2}c_2=x_i-p_{s_i,1}c_1$ i.e. $c_2=[x_i-p_{s_i,1}c_1]/p_{s_i,2}$.

$$\begin{aligned} &f(\mathbf{x},\boldsymbol{\theta};a,\mathbf{a}_0,\boldsymbol{\nu},\mathbf{p}) \\ &= \Big[\prod_{i=1}^n\big(\int_{\mathbb{R}}\int_{\mathbb{R}}\big(f_{\Gamma(a,\theta_1)}(c_1)f_{\Gamma(a,\theta_2)}(c_2)\big)1_{\{p_{s_i,1}c_1+p_{s_i,2}c_2=x_i\}\cap\{c_t\geq 0\}}dc_1\,dc_2\big)\Big]\Big[f_{\Gamma(a_{0,1},\nu_1)}(\theta_1)f_{\Gamma(a_{0,2},\nu_2)}(\theta_2)\Big] \\ &= \Big[\prod_{i=1}^n\big(\int_0^{x_i/p_{s_i,1}}\big(f_{\Gamma(a,\theta_1)}(c_1)f_{\Gamma(a,\theta_2)}([x_i-p_{s_i,1}c_1]/p_{s_i,2})\big)dc_1\big)\Big]\Big[f_{\Gamma(a_{0,1},\nu_1)}(\theta_1)f_{\Gamma(a_{0,2},\nu_2)}(\theta_2)\Big] \\ &= \prod_{i=1}^n\big(\int_0^{x_i/p_{s_i,1}}\big(f_{\Gamma(a,\theta_1)}(c_1)f_{\Gamma(a,\theta_2)}([x_i-p_{s_i,1}c_1]/p_{s_i,2})\big)dc_1[f_{\Gamma(a_{0,1},\nu_1)}(\theta_1)f_{\Gamma(a_{0,2},\nu_2)}(\theta_2)]^{1/n}\big) \end{aligned}$$

We can evaluate this using

```
fGamma := (a,theta,c) -> (theta^a) * (c^(a-1)) * exp(-c*theta) / GAMMA(a);
upperLimit := x/p1;
integrand := fGamma(a,theta1,c1) * fGamma(a,theta2,(x-p1*c1)/p2);
integral := int(integrand,c1=0..upperLimit) assuming a>0,theta1>0,theta2>0,
prior := (fGamma(a01,nu1,theta1)*fGamma(a02,nu2,theta2))^(1/n);
wholeTerm := integral*prior;
```

but there isn't a closed form. So we will use numerical integration to check it.

Next, still for $T = 2$,

$$f(\mathbf{x}, \text{च}; a, \mathbf{a}_0, \boldsymbol{\nu}, \mathbf{p})$$
$$= \Big[2^n \prod_{i=1}^{n} \Big(\frac{\Gamma(a+a_{0,1})}{\Gamma(a)\Gamma(a_{0,1})} \frac{\Gamma(a+a_{0,2})}{\Gamma(a)\Gamma(a_{0,2})} \frac{(c_{i,1}x_i/p_{s_i,1})^{a-1}\nu_1^{a_{0,1}}}{((c_{i,1}x_i/p_{s_i,1})+\nu_1)^{a+a_{0,1}}} \frac{(c_{i,2}x_i/p_{s_i,2})^{a-1}\nu_2^{a_{0,2}}}{((c_{i,2}x_i/p_{s_i,2})+\nu_2)^{a+a_{0,2}}}\Big)\Big] \Big[\frac{1}{2^n} \prod_{i=1}^{n} 1_{\{\sum_{t=1}^{T} c_{i,t}=1\}\cap\{c_{i,t} \geq$$

HERE

EM algorithm

Now we want to find an EM algorithm to learn the parameters. Note that the parameters are $\boldsymbol{\alpha} = (a, a_{0,t}, \nu_t, p_{1,t}, \ldots, p_{4,t})_{t=1}^{T}$.

We follow Wasserman. We start with an initial set of parameters $\boldsymbol{\alpha}^0$, and then for $j = 0, 1, 2, \ldots$ we repeat the E step and the M step.

The E step is to calculate

$$J(\boldsymbol{\alpha}|\boldsymbol{\alpha}^j) = E\Big(\log \frac{f(\mathbf{x}, \text{छ}; \boldsymbol{\alpha})}{f(\mathbf{x}, \text{छ}; \boldsymbol{\alpha}^j)}\Big)$$

where the expectation is over छ (see Step 8).

The M step is to find $\boldsymbol{\alpha}^{j+1}$ that maximizes $\boldsymbol{\alpha} \mapsto J(\boldsymbol{\alpha}|\boldsymbol{\alpha}^j)$.

E step

Begin tmp:

$$\begin{aligned}
f_{\text{cond}}(\mathbf{x}|\text{च}; a, \mathbf{a}_0, \nu, \mathbf{p}) &= (T!)^n \prod_{i=1}^{n} f_{\text{post}}((c_{i,t}x/p_{s_i,t})_{t=1}^{T}; a, \mathbf{a}_0, \boldsymbol{\nu}) \\
f_{\text{prior}}(\text{च}) &= \frac{1}{(T!)^n} \prod_{i=1}^{n} 1_{\Delta_T}(\mathbf{c}^{(i)}) \\
&= \frac{1}{(T!)^n} \prod_{i=1}^{n} 1_{\{\sum_{t=1}^{T} c_{i,t}=1\} \cap \{c_{i,t} \geq 0\}} \\
f_{\text{post}}(\mathbf{c}^{(i)}; a, \mathbf{a}_0, \boldsymbol{\nu}) &= \prod_{t=1}^{T} \frac{\Gamma(a + a_{0,t})}{\Gamma(a)\Gamma(a_{0,t})} \frac{c_{i,t}^{a-1} \nu_t^{a_{0,t}}}{(c_{i,t} + \nu_t)^{a+a_{0,t}}}
\end{aligned}$$

End tmp.

We compute $J(\boldsymbol{\alpha}|\boldsymbol{\alpha}^j)$ in more detail. Note

$$
\begin{aligned}
&\log f(\mathbf{x}, \text{च}; \boldsymbol{\alpha}) \\
&= \log f_{\text{cond}}(\mathbf{x}|\text{च}; a, \mathbf{a}_0, \nu, \mathbf{p}) + \log f_{\text{prior}}(\text{च}) \\
&= \log \left[(T!)^n \prod_{i=1}^{n} f_{\text{post}}((c_{i,t} x_i / p_{s_i,t})_{t=1}^{T}; a, \mathbf{a}_0, \boldsymbol{\nu}) \right] + \log f_{\text{prior}}(\text{च}) \\
&= \log \left[(T!)^n \prod_{i=1}^{n} \prod_{t=1}^{T} \frac{\Gamma(a + a_{0,t})}{\Gamma(a)\Gamma(a_{0,t})} \frac{(c_{i,t} x_i / p_{s_i,t})^{a-1} \nu_t^{a_{0,t}}}{((c_{i,t} x_i / p_{s_i,t}) + \nu_t)^{a + a_{0,t}}} \right] + \log f_{\text{prior}}(\text{च}) \\
&= n \log(T!) + \sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log \Gamma(a + a_{0,t}) - \log \Gamma(a) - \log \Gamma(a_{0,t}) \\
&\quad + (a-1)[\log c_{i,t} + \log x_i - \log p_{s_i,t}] + a_{0,t} \log \nu_t \\
&\quad - (a + a_{0,t}) \log((c_{i,t} x_i / p_{s_i,t}) + \nu_t) \Big] + \log f_{\text{prior}}(\text{च})
\end{aligned}
$$

So (using $\tilde{\alpha} = \alpha^j$)

$$
\begin{aligned}
&\log \frac{f(\mathbf{x}, \text{च}; \boldsymbol{\alpha})}{f(\mathbf{x}, \text{च}; \tilde{\boldsymbol{\alpha}})} \\
&= \Bigg(n \log(T!) + \sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log \Gamma(a + a_{0,t}) - \log \Gamma(a) - \log \Gamma(a_{0,t}) \\
&\quad + (a-1)[\log c_{i,t} + \log x_i - \log p_{s_i,t}] + a_{0,t} \log \nu_t \\
&\quad - (a + a_{0,t}) \log((c_{i,t} x_i / p_{s_i,t}) + \nu_t) \Big] + \log f_{\text{prior}}(\text{च}) \Bigg) \\
&\quad - \Bigg(n \log(T!) + \sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log \Gamma(\tilde{a} + \tilde{a}_{0,t}) - \log \Gamma(\tilde{a}) - \log \Gamma(\tilde{a}_{0,t}) \\
&\quad + (\tilde{a}-1)[\log c_{i,t} + \log x_i - \log \tilde{p}_{s_i,t}] + \tilde{a}_{0,t} \log \tilde{\nu}_t \\
&\quad - (\tilde{a} + \tilde{a}_{0,t}) \log((c_{i,t} x_i / \tilde{p}_{s_i,t}) + \tilde{\nu}_t) \Big] + \log f_{\text{prior}}(\text{च}) \Bigg) \\
&= \Bigg(\sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log \Gamma(a + a_{0,t}) - \log \Gamma(a) - \log \Gamma(a_{0,t}) \\
&\quad + (a-1)[\log c_{i,t} + \log x_i - \log p_{s_i,t}] + a_{0,t} \log \nu_t \\
&\quad - (a + a_{0,t}) \log((c_{i,t} x_i / p_{s_i,t}) + \nu_t) \Big] \Bigg) \\
&\quad - \Bigg(\sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log \Gamma(\tilde{a} + \tilde{a}_{0,t}) - \log \Gamma(\tilde{a}) - \log \Gamma(\tilde{a}_{0,t}) \\
&\quad + (\tilde{a}-1)[\log c_{i,t} + \log x_i - \log \tilde{p}_{s_i,t}] + \tilde{a}_{0,t} \log \tilde{\nu}_t \\
&\quad - (\tilde{a} + \tilde{a}_{0,t}) \log((c_{i,t} x_i / \tilde{p}_{s_i,t}) + \tilde{\nu}_t) \Big] \Bigg)
\end{aligned}
$$

Next, we take the expectation. Note that f_{prior} is a probability density on $\mathbb{R}^{n\times T}$ so we can simply pull out terms that do not involve च.

$$
\begin{aligned}
&E \log \frac{f(\mathbf{x}, \text{छ}; \boldsymbol{\alpha})}{f(\mathbf{x}, \text{छ}; \tilde{\boldsymbol{\alpha}})} \\
&= \int_{\mathbb{R}^{n\times T}} f_{\text{prior}}(\text{च}) \log \frac{f(\mathbf{x}, \text{च}; \boldsymbol{\alpha})}{f(\mathbf{x}, \text{च}; \tilde{\boldsymbol{\alpha}})} \, d\text{च} \\
&= \Bigg(\sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log\Gamma(a + a_{0,t}) - \log\Gamma(a) - \log\Gamma(a_{0,t}) \\
&\qquad + (a-1)[\textstyle\int_{\mathbb{R}^{n\times T}} \big(f_{\text{prior}}(\text{च}) \log c_{i,t}\big) \, d\text{च} + \log x_i - \log p_{s_i,t}] + a_{0,t} \log \nu_t \\
&\qquad - (a + a_{0,t}) \textstyle\int_{\mathbb{R}^{n\times T}} \big(f_{\text{prior}}(\text{च}) \log((c_{i,t} x_i / p_{s_i,t}) + \nu_t)\big) \, d\text{च} \Big] \Bigg) \\
&\quad - \Bigg(\sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log\Gamma(\tilde{a} + \tilde{a}_{0,t}) - \log\Gamma(\tilde{a}) - \log\Gamma(\tilde{a}_{0,t}) \\
&\qquad + (\tilde{a}-1)[\textstyle\int_{\mathbb{R}^{n\times T}} \big(f_{\text{prior}}(\text{च}) \log c_{i,t}\big) \, d\text{च} + \log x_i - \log \tilde{p}_{s_i,t}] + \tilde{a}_{0,t} \log \tilde{\nu}_t \\
&\qquad - (\tilde{a} + \tilde{a}_{0,t}) \textstyle\int_{\mathbb{R}^{n\times T}} \big(f_{\text{prior}}(\text{च}) \log((c_{i,t} x_i / \tilde{p}_{s_i,t}) + \tilde{\nu}_t)\big) \, d\text{च} \Big] \Bigg)
\end{aligned}
$$

Before continuing, it would be convenient to evaluate or simplify the integrals here. There are two forms of integrals:

$$
\begin{aligned}
I_1 &\stackrel{\text{def}}{=} \textstyle\int_{\mathbb{R}^{n\times T}} \big(f_{\text{prior}}(\text{च}) \log c_{i,t}\big) \, d\text{च} \quad \text{and} \\
I_2(x_i, p_{s_i,t}, \nu_t) &\stackrel{\text{def}}{=} \textstyle\int_{\mathbb{R}^{n\times T}} \big(f_{\text{prior}}(\text{च}) \log((c_{i,t} x_i / p_{s_i,t}) + \nu_t)\big) \, d\text{च}
\end{aligned}
$$

These integrals are evaluated in "Appendix on $\int_{\mathbb{R}^{n\times T}} \big(f_{\text{prior}}(\text{च}) \log c_{i,t}\big) \, d\text{च}$" and "Appendix on $\int_{\mathbb{R}^{n\times T}} \big(f_{\text{prior}}(\text{च}) \log((c$
$\nu_t)\big) \, d\text{च}$" below. The result is:

$$
\begin{aligned}
I_1 &= -\frac{H_T}{(T!)^2} \\
I_2(x_i, p_{s_i,t}, \nu_t) &= \frac{1}{T!(T-1)!} \int_0^1 \log((c_{i,t} x_i / p_{s_i,t}) + \nu_t)(1 - c_{i,t})^{T-1} dc_{i,t}
\end{aligned}
$$

where H_T is the Tth harmonic number. So far, I haven't been able to find a closed form for I_2.

Plugging in, we get

$$
\begin{aligned}
&E \log \frac{f(\mathbf{x}, \text{छग}; \boldsymbol{\alpha})}{f(\mathbf{x}, \text{छग}; \tilde{\boldsymbol{\alpha}})} \\
&= \Bigg(\sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log \Gamma(a + a_{0,t}) - \log \Gamma(a) - \log \Gamma(a_{0,t}) \\
&\quad + (a-1)[I_1 + \log x_i - \log p_{s_i,t}] + a_{0,t} \log \nu_t \\
&\quad - (a + a_{0,t}) I_2(x_i, p_{s_i,t}, \nu_t) \Big] \Bigg) \\
&\quad - \Bigg(\sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log \Gamma(\tilde{a} + \tilde{a}_{0,t}) - \log \Gamma(\tilde{a}) - \log \Gamma(\tilde{a}_{0,t}) \\
&\quad + (\tilde{a}-1)[I_1 + \log x_i - \log \tilde{p}_{s_i,t}] + \tilde{a}_{0,t} \log \tilde{\nu}_t \\
&\quad - (\tilde{a} + \tilde{a}_{0,t}) I_2(x_i, \tilde{p}_{s_i,t}, \tilde{\nu}_t) \Big] \Bigg)
\end{aligned}
$$

Next, we drop terms that are constant with respect to $\boldsymbol{\alpha}$. Note that this includes actually the entire denominator, because the prior distribution does not depend on $\boldsymbol{\alpha}$ at all:

$$
\begin{aligned}
&E \log \frac{f(\mathbf{x}, \text{छग}; \boldsymbol{\alpha})}{f(\mathbf{x}, \text{छग}; \tilde{\boldsymbol{\alpha}})} \\
&= \Bigg(\sum_{i=1}^{n} \sum_{t=1}^{T} \Big[\log \Gamma(a + a_{0,t}) - \log \Gamma(a) - \log \Gamma(a_{0,t}) \\
&\quad + (a-1)[I_1 + \log x_i - \log p_{s_i,t}] + a_{0,t} \log \nu_t \\
&\quad - (a + a_{0,t}) I_2(x_i, p_{s_i,t}, \nu_t) \Big] \Bigg)
\end{aligned}
$$

Is something wrong...?

Appendix on simplifying $f_{\text{one}(i)}(x)$

(This material originally followed Step 3. I think that the calculations here are correct, but they don't go anywhere, so they aren't used anywhere else right now.)

We would like to rewrite this density so that it is easier to evaluate on a computer. Substituting $c'_t = c_t p_t / x$, we get

$$f_{\text{one}(i)}(x) = \int_{\mathbb{R}^T} \prod_{t=1}^{T} \big(f_{\Gamma(a,\theta_t)}(c'_t x / p_t) \big) 1_{\{\sum_{t=1}^T c'_t = 1\} \cap \{c_t \geq 0\}} d\mathbf{c}'$$

Recall that the Gamma density satisfies

$$f_{\Gamma(a,\theta)}(cx) = (1/x) f_{\Gamma(a,\theta x)}(c)$$

so

$$\begin{aligned}
f_{\text{one}(i)}(x) &= \int_{\mathbb{R}^T} \prod_{t=1}^{T} (p_t/x) \big(f_{\Gamma(a,\theta_t x/p_t)}(c'_t) \big) 1_{\{\sum_{t=1}^T c'_t = 1\} \cap \{c'_t \geq 0\}} d\mathbf{c}' \\
&= \Big(\prod_{t=1}^{T} p_t / x \Big) \int_{\mathbb{R}^T} \prod_{t=1}^{T} \big(f_{\Gamma(a,\theta_t x/p_t)}(c'_t) \big) 1_{\{\sum_{t=1}^T c'_t = 1\} \cap \{c'_t \geq 0\}} d\mathbf{c}' \\
&= \Big(\prod_{t=1}^{T} p_t / x \Big) f_{\text{inner}(i)}(x)
\end{aligned}$$

where

$$f_{\text{inner}(i)}(x) = \int_{\mathbb{R}^T} \prod_{t=1}^{T} \big(f_{\Gamma(a,\theta_t x/p_t)}(c'_t) \big) 1_{\{\sum_{t=1}^T c'_t = 1\} \cap \{c'_t \geq 0\}} d\mathbf{c}'$$

Note that $f_{\text{inner}(i)}(x)$ is an integral of a product of densities over a probability simplex. We can therefore compute it via a Laplace transform as described in Wolpert and Wolf (1995), as follows. First, note that the Laplace transform of $f_{\Gamma(a,\theta)}(c)$ is (Wikipedia)

$$\mathscr{L}_{\Gamma(a,\theta)}(s) = \frac{\theta^a}{(s+\theta)^a}$$

So

$$\begin{aligned}
\mathscr{L}_{\Gamma(a,\theta_t x/p_t)}(s) &= \frac{(\theta_t x/p_t)^a}{(s + (\theta_t x/p_t))^a} \\
&= \frac{(\theta_t x/p_t)^a}{((s p_t + \theta_t x)/p_t)^a} \\
&= \frac{(\theta_t x)^a}{(s p_t + \theta_t x)^a} \\
&= \frac{1}{(s p_t/(\theta_t x) + 1)^a} \\
&= (1 + \tfrac{s p_t}{\theta_t x})^{-a}
\end{aligned}$$

According to the Laplace convolution theorem (Theorem 2 in Wolpert and Wolf 1995), if $\otimes$ denotes the Laplace convolution operator defined by

$$(f \otimes g)(\tau) = \int_0^\tau f(\tau - t) g(t)\, dt$$

then

$$\mathscr{L}(\otimes_{t=1}^T f_t)(s) = \prod_{t=1}^T \mathscr{L}(f_t)(s)$$

So specifically

$$\begin{aligned}
\mathscr{L}(f_{\text{inner}(i)})(s) &= \mathscr{L}(\otimes_{t=1}^T f_{\Gamma(a,\theta_t x/p_t)})(s) \\
&= \prod_{t=1}^T \mathscr{L}_{\Gamma(a,\theta_t x/p_t)}(s) \\
&= \prod_{t=1}^T (1 + \tfrac{s p_t}{\theta_t x})^{-a}
\end{aligned}$$

To get $f_{\text{inner}(i)}$, we take the inverse Laplace transform and evaluate it at $\tau = 1$ (Wolpert and Wolf 1995, Theorem 1):

$$\begin{aligned}
f_{\text{inner}(i)}(x) &= (\mathscr{L}^{-1}\mathscr{L}(f_{\text{inner}(i)}))(1) \\
&= \Big(\mathscr{L}^{-1}\Big[\prod_{t=1}^T (1 + \tfrac{s p_t}{\theta_t x})^{-a}\Big]\Big)(1) \\
&= \Big(\frac{1}{2\pi i}\lim_{u\to\infty}\int_{\gamma-iu}^{\gamma+iu} e^{s\tau}\prod_{t=1}^T (1 + \tfrac{s p_t}{\theta_t x})^{-a}\, ds\Big)(\tau = 1) \\
&= \frac{1}{2\pi i}\lim_{u\to\infty}\int_{\gamma-iu}^{\gamma+iu} e^{s}\prod_{t=1}^T (1 + \tfrac{s p_t}{\theta_t x})^{-a}\, ds
\end{aligned}$$

where "γ is a real number so that the contour path of integration is in the region of convergence of" $\mathscr{L}(f_{\text{inner}(i)})$ (Wikipedia). In particular (Wikipedia) choose γ so that $\gamma >$Re σ for all singularities σ of $\mathscr{L}(f_{\text{inner}(i)})$. In our case, $\mathscr{L}(f_{\text{inner}(i)})$ doesn't have any singularities if $\gamma \geq 0$, since $p_t/(\theta_t x) \geq 0$ and $a > 0$. I guess that there are uncountably many singularities if $\gamma < 0$. So let's choose $\gamma = 0$:

$$f_{\text{inner}(i)}(x) = \frac{1}{2\pi i}\lim_{u\to\infty}\int_{-iu}^{iu} e^{s}\prod_{t=1}^T (1 + \tfrac{s p_t}{\theta_t x})^{-a}\, ds$$

Note that

$$\begin{aligned}
1 + \frac{s p_t}{\theta_t x} &= 1 + \frac{s}{\theta_t x/p_t} \\
&= \frac{(\theta_t x/p_t) + s}{\theta_t x/p_t}
\end{aligned}$$

so

$$f_{\text{inner}(i)}(x) = \frac{1}{2\pi i} \lim_{u\to\infty} \int_{-iu}^{iu} e^s \prod_{t=1}^{T} \frac{((\theta_t x/p_t)+s)^{-a}}{(\theta_t x/p_t)^{-a}}\, ds$$

$$= \frac{\prod_{t=1}^{T} (\theta_t x/p_t)^a}{2\pi i} \lim_{u\to\infty} \int_{-iu}^{iu} e^s \prod_{t=1}^{T} ((\theta_t x/p_t)+s)^{-a}\, ds$$

$$= \frac{\prod_{t=1}^{T} (\theta_t x/p_t)^a}{2\pi i} \mathscr{L}^{-1}_{\text{blah}}((\theta_t x/p_t)_{t=1}^{T})$$

where

$$\mathscr{L}^{-1}_{\text{blah}}(\mathbf{q}) = \lim_{u\to\infty} \int_{-iu}^{iu} e^s \prod_{t=1}^{T} (q_t+s)^{-a}\, ds$$

[Note that in general[1] for $\nu > 0$, if the Laplace transform is

$$\mathscr{L}(p) = ((p+a)(p+b))^{-\nu}$$

then the inverse Laplace transform of it is

$$f(x) = \frac{\sqrt{\pi}}{\Gamma(\nu)} \big(\frac{x}{a-b}\big)^{\nu-1/2} \exp\big(-\frac{a+b}{2}x\big) I_{\nu-1/2}\big(\frac{a-b}{2}x\big)$$

where $I_\nu(x)$ is the modified Bessel function of the first kind.]

Another way to approach $\mathscr{L}^{-1}_{\text{blah}}$ is to use the following fact[2]: If the Laplace transform is $g_1(s)g_2(s)$ then the inverse Laplace transform is $(g_1 \otimes g_2)(\tau)$. So in general, if the Laplace transform is $\prod_{t=1}^{T} g_t(s)$ then the inverse Laplace transform is $(\otimes_{t=1}^{T} g_t)(\tau)$. In our case above the Laplace transform is

$$\prod_{t=1}^{T} (q_t+s)^{-a}$$

so the inverse Laplace transform is

$$(\otimes_{t=1}^{T} (s \mapsto (q_t+s)^{-a})))(\tau)$$

It seems that we need to give up on this for now.

[1] `http://eqworld.ipmnet.ru/en/auxiliary/inttrans/LapInv4.pdf`, No. 3
[2] `http://eqworld.ipmnet.ru/en/auxiliary/inttrans/LapInv1.pdf`, No. 15

Appendix on volumes of simplices

The material in this section is based mostly on `http://www.math.niu.edu/~rusin/known-math/97/volumes.polyh` and Wikipedia s.v. "Simplex".

The standard simplex has $(T+1)$ vertices $0, e_1, e_2, \ldots, e_T \in \mathbb{R}^T$. Let $w_1 = e_1 - 0, w_2 = e_2 - 0, \ldots, w_T = e_T - 0$. Let W be the $T \times T$ matrix whose rows are the row vectors w_t. Then the Gram determinant formula says that the volume of our simplex is

$$\frac{|\det WW'|^{1/2}}{T!}$$

Note that since W is the $T \times T$ identity matrix, $\det WW' = \det I_T = 1$ so the volume of our simplex is $1/T!$, which agrees with Wikipedia. (Compare the section on 'Probability" and the section on 'Geometric properties" in the article on 'Simplex".)

Note also that if our simplex has $(T+1)$ vertices $0, a_1 e_1, \ldots, a_T e_T \in \mathbb{R}^T$, where all the a_t are strictly positive, then as above the volume of our simplex is

$$\frac{|\det WW'|^{1/2}}{T!}$$

where W is a diagonal matrix with (t,t) entry a_1. So the volume is

$$\begin{aligned} \frac{|\det WW'|^{1/2}}{T!} &= \frac{|\det W|}{T!} \\ &= \frac{\prod_{t=1}^{T} a_t}{T!} \end{aligned}$$

Appendix on $\int_{\mathbb{R}^{n\times T}} \left(f_{\text{prior}}(\text{च}) \log c_{i,t}\right) d\text{च}$

First, note that

$$
\begin{aligned}
I &\stackrel{\text{def}}{=} \int_{\mathbb{R}^{n\times T}} \left(f_{\text{prior}}(\text{च}) \log c_{i,t}\right) d\text{च} \\
&= \int_{\mathbb{R}^{n\times T}} \left(\frac{1}{(T!)^n} \prod_{i=1}^{n} 1_{\{\sum_{t=1}^{T} c_{i,t}=1\}\cap\{c_{i,t}\geq 0\}}\right) \left(\log c_{i,t}\right) d\text{च} \\
&= \frac{1}{T!} \int_{\mathbb{R}^{T}} \left(1_{\{\sum_{t=1}^{T} c_{i,t}=1\}\cap\{c_{i,t}\geq 0\}}\right) (\log c_{i,t})\, d\mathbf{c}^{(i)}
\end{aligned}
$$

due to the normalization $1/T!$ for each n. Next, note that (dropping the i index for now), if the integrand is $\log c_1$, we get

$$
\begin{aligned}
\tilde{I} &\stackrel{\text{def}}{=} \int_{\mathbb{R}^{T}} \left(1_{\{\sum_{t=1}^{T} c_t=1\}\cap\{c_t\geq 0\}}\right) (\log c_1)\, d\mathbf{c} \\
&= \int_0^1 \int_0^{1-c_1} \int_0^{1-c_1-c_2} \cdots \int_0^{1-\sum_{t=1}^{T-1} c_t} (\log c_1)\, dc_T\, dc_{T-1} \cdots dc_1 \\
&= \int_0^1 (\log c_1) \left[\int_0^{(1-c_1)} \int_0^{(1-c_1)-c_2} \cdots \int_0^{(1-c_1)-\sum_{t=2}^{T-1} c_t} dc_T\, dc_{T-1} \cdots dc_2\right] dc_1 \\
&= \int_0^1 (\log c_1) V_{T-1}(1-c_1) dc_1
\end{aligned}
$$

where $V_{T-1}(a)$ is the volume of a simplex in $\mathbb{R}^{T-1}$ with T vertices $0, ae_1, ae_2, \ldots, ae_{T-1} \in \mathbb{R}^{T-1}$. From "Appendix on volumes of simplices", we know that

$$
V_{T-1}(a) = \frac{\prod_{t=1}^{T-1} a}{(T-1)!} = \frac{a^{T-1}}{(T-1)!}
$$

So

$$
\begin{aligned}
\tilde{I} &= \int_0^1 (\log c_1) \frac{(1-c_1)^{T-1}}{(T-1)!} dc_1 \\
&= -\frac{\gamma T + \Psi(T) T + 1}{T^2 (T-1)!} \quad \text{(Maple)} \\
&= -\frac{H_T}{T(T-1)!} \quad \text{(Wolfram Alpha)} \\
&= -\frac{H_T}{T!}
\end{aligned}
$$

where γ is Euler's constant, $\Psi(x)$ is digamma function, and H_T is the Tth harmonic number $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{T}$. (Note: Maple and WolframAlpha agree, since according to Wikipedia $\Psi(n) = H_{n-1} - \gamma$, so $\frac{\gamma T + \Psi(T) T + 1}{T^2 (T-1)!} = \frac{\gamma + \Psi(T) + \frac{1}{T}}{T(T-1)!} = \frac{H_{T-1} + \frac{1}{T}}{T!} = \frac{H_T}{T!}$.)

By symmetry (since by Fubini we can perform the integration coordinatewise in any order), the above holds also for integrands $\log c_t$ when t is bigger than 1.

Thus, plugging in, we get

$$
\begin{aligned}
I &= \frac{1}{T!}\int_{\mathbb{R}^T}\left(1_{\{\sum_{t=1}^{T} c_{i,t}=1\}\cap\{c_{i,t}\geq 0\}}\right)(\log c_{i,t})\, d\mathbf{c}^{(i)} \\
&= -\frac{1}{T!}\frac{H_T}{T!} \\
&= -\frac{H_T}{(T!)^2}
\end{aligned}
$$

Appendix on $\int_{\mathbb{R}^{n\times T}} \left(f_{\text{prior}}(\text{च}) \log((c_{i,t}x_i/p_{s_i,t}) + \nu_t)\right) d\text{च}$

We approach this similarly to the previous integral. First, note that

$$\begin{aligned} I &\stackrel{\text{def}}{=} \int_{\mathbb{R}^{n\times T}} \left(f_{\text{prior}}(\text{च}) \log((c_{i,t}x_i/p_{s_i,t}) + \nu_t)\right) d\text{च} \\ &= \frac{1}{T!} \int_{\mathbb{R}^T} \left(1_{\{\sum_{t=1}^T c_{i,t}=1\}\cap\{c_{i,t}\geq 0\}}\right) \log((c_{i,t}x_i/p_{s_i,t}) + \nu_t)\, d\mathbf{c}^{(i)} \end{aligned}$$

Dropping the index i for now, if the integrand is $\log((c_1x/p_1)+\nu_1)$, we get

$$\begin{aligned} \tilde{I} &\stackrel{\text{def}}{=} \int_{\mathbb{R}^T} \left(1_{\{\sum_{t=1}^T c_t=1\}\cap\{c_t\geq 0\}}\right) \log((c_1x/p_1) + \nu_1)\, d\mathbf{c} \\ &= \int_0^1 \log((c_1x/p_1)+\nu_1) V_{T-1}(1-c_1) dc_1 \\ &= \int_0^1 \log((c_1x/p_1)+\nu_1) \frac{(1-c_1)^{T-1}}{(T-1)!} dc_1 \\ &= \frac{1}{(T-1)!}\left[\frac{\log \nu_1}{T} + \Gamma(T)\frac{\nu_1 p_1}{x}\text{MeijerG}\left([[-1],[T]],[[-1,-1],[]],\frac{\nu_1 p_1}{x}\right)\right] \quad \text{(Maple)} \\ &= \frac{\log \nu_1}{T!} + \frac{\nu_1 p_1}{x}\text{MeijerG}\left([[-1],[T]],[[-1,-1],[]],\frac{\nu_1 p_1}{x}\right) \end{aligned}$$

where MeijerG is the inverse Laplace transform

$$\text{MeijerG}([\mathbf{a},\mathbf{b}],[\mathbf{c},\mathbf{d}],z) = \frac{1}{2\pi i}\oint_L \frac{(\prod_{i=1}^{n_\mathbf{a}}\Gamma(1-a_i+y))(\prod_{i=1}^{n_\mathbf{c}}\Gamma(c_i-y))}{(\prod_{i=1}^{n_\mathbf{b}}\Gamma(b_i-y))(\prod_{i=1}^{n_\mathbf{d}}\Gamma(1-d_i+y))} z^y\, dy$$

and L is the contour from $\gamma - i\infty$ to $\gamma + i\infty$ for a particular γ. So in our particular case

$$\begin{aligned} &\text{MeijerG}\left([[-1],[T]],[[-1,-1],[]],\frac{\nu_1 p_1}{x}\right) \\ &= \frac{1}{2\pi i}\oint_L \frac{(\Gamma(1-(-1)+y))(\Gamma(-1-y)^2)}{(\Gamma(T-y))(1)} \left(\frac{\nu_1 p_1}{x}\right)^y dy \\ &= \frac{1}{2\pi i}\oint_L \frac{\Gamma(2+y)\Gamma(-1-y)^2}{\Gamma(T-y)} \left(\frac{\nu_1 p_1}{x}\right)^y dy \end{aligned}$$

Anyhow, this is way too complicated and we need to figure something else out.

For now, we will just leave the overall integral as

$$\begin{aligned} I &= \frac{1}{T!}\int_0^1 \log((c_{i,t}x_i/p_{s_i,t})+\nu_t)\frac{(1-c_{i,t})^{T-1}}{(T-1)!} dc_{i,t} \\ &= \frac{1}{T!(T-1)!}\int_0^1 \log((c_{i,t}x_i/p_{s_i,t})+\nu_t)(1-c_{i,t})^{T-1} dc_{i,t} \end{aligned}$$