The Quest for Faster Join Algorithms

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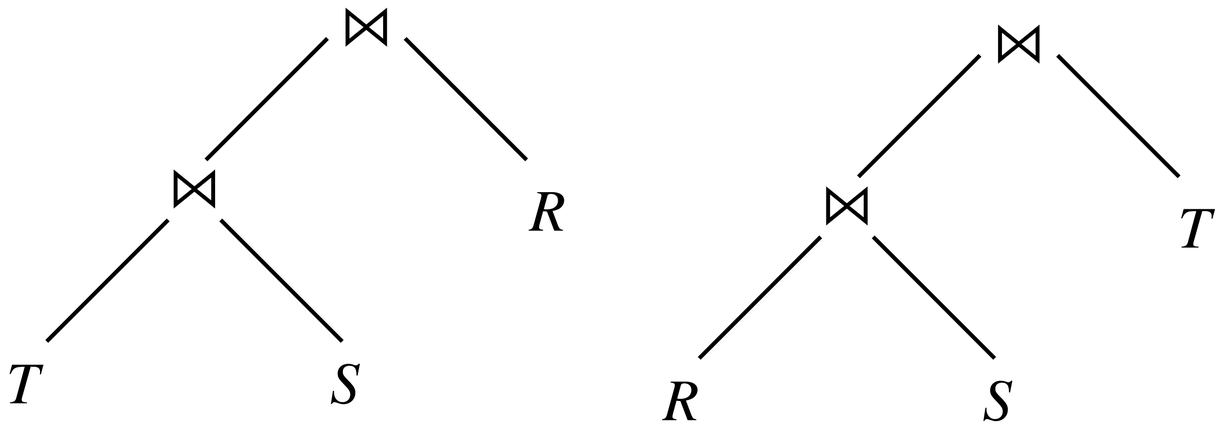
Joins are a core relational operator

The Long History of Join Algorithms

- typically computed via a query plan of **binary join operators**
- many different algorithms to compute binary joins
  - nested loop join
  - hash join
  - sort-merge join
  - .. and a lot more!

$$R(A, B) \bowtie S(B, C) \bowtie T(C, D)$$



But haven’t we already solved everything?

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No!

1. Query optimizers are notoriously sensitive to cardinality estimation errors leading to suboptimal plans
2. Binary join plans perform well for most joins, but poorly for others:
   - pattern matching on graphs
   - cyclic joins
3. There is a large gap between theory and practice!

What is the fastest algorithm to compute a join query?

We still don’t know!

What is the fastest algorithm to compute a join query?

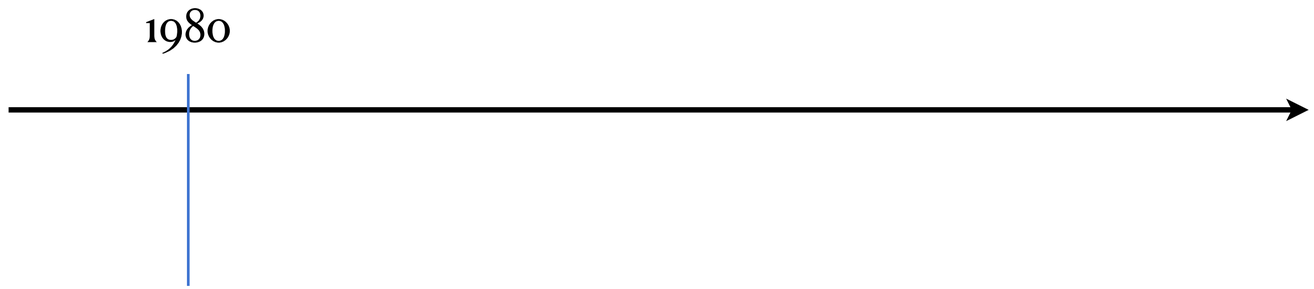
But we have made some amazing progress!

In this talk:

the story about the progress and a few key ideas behind it

Roadmap

Roadmap



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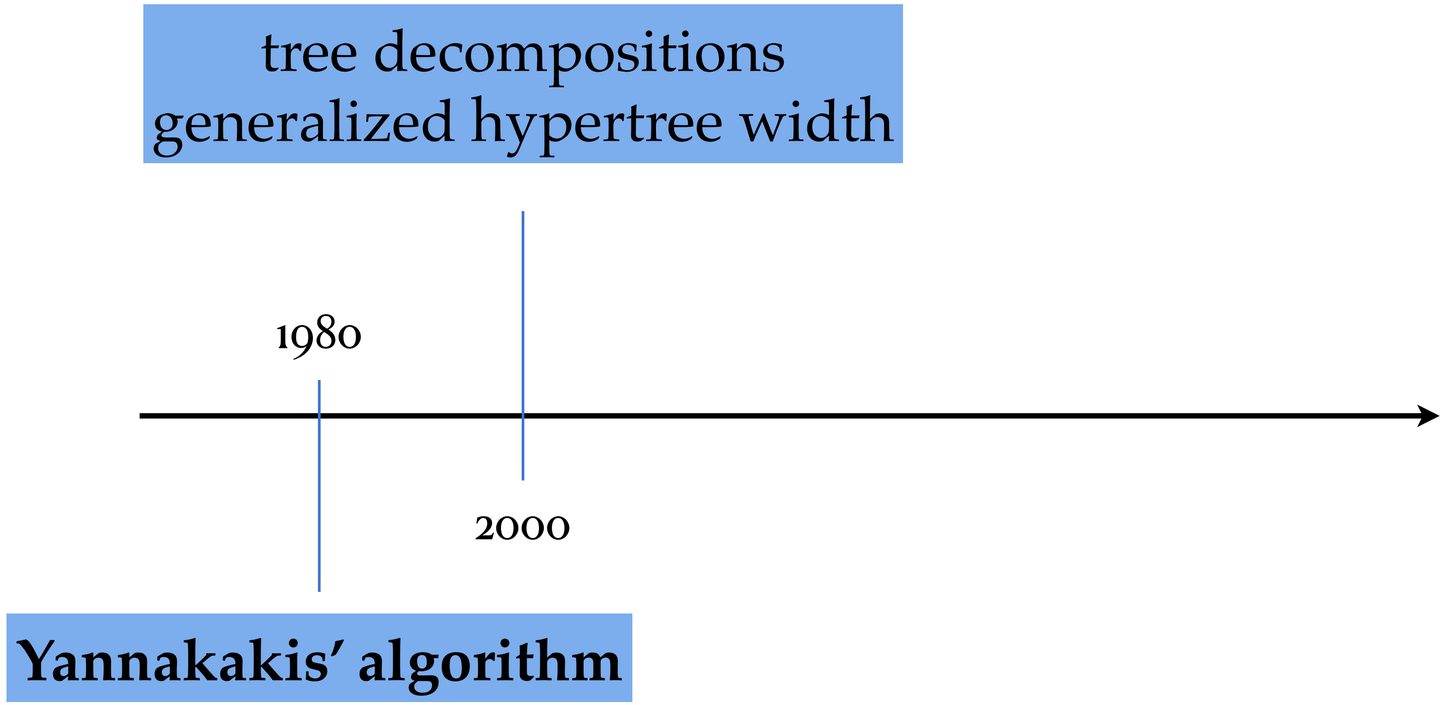
tree decompositions
generalized hypertree width
1980
2000
Yannakakis' algorithm

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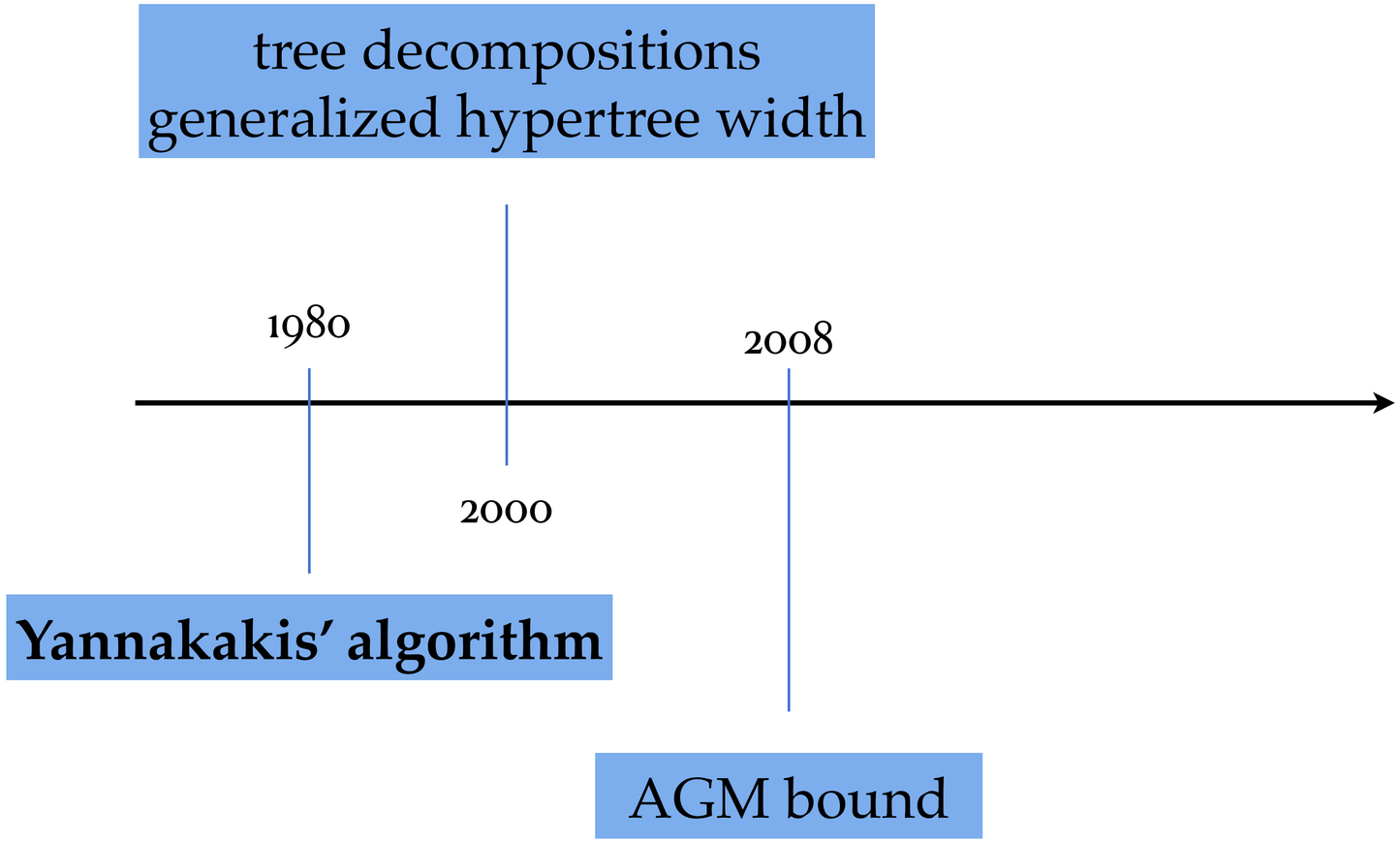
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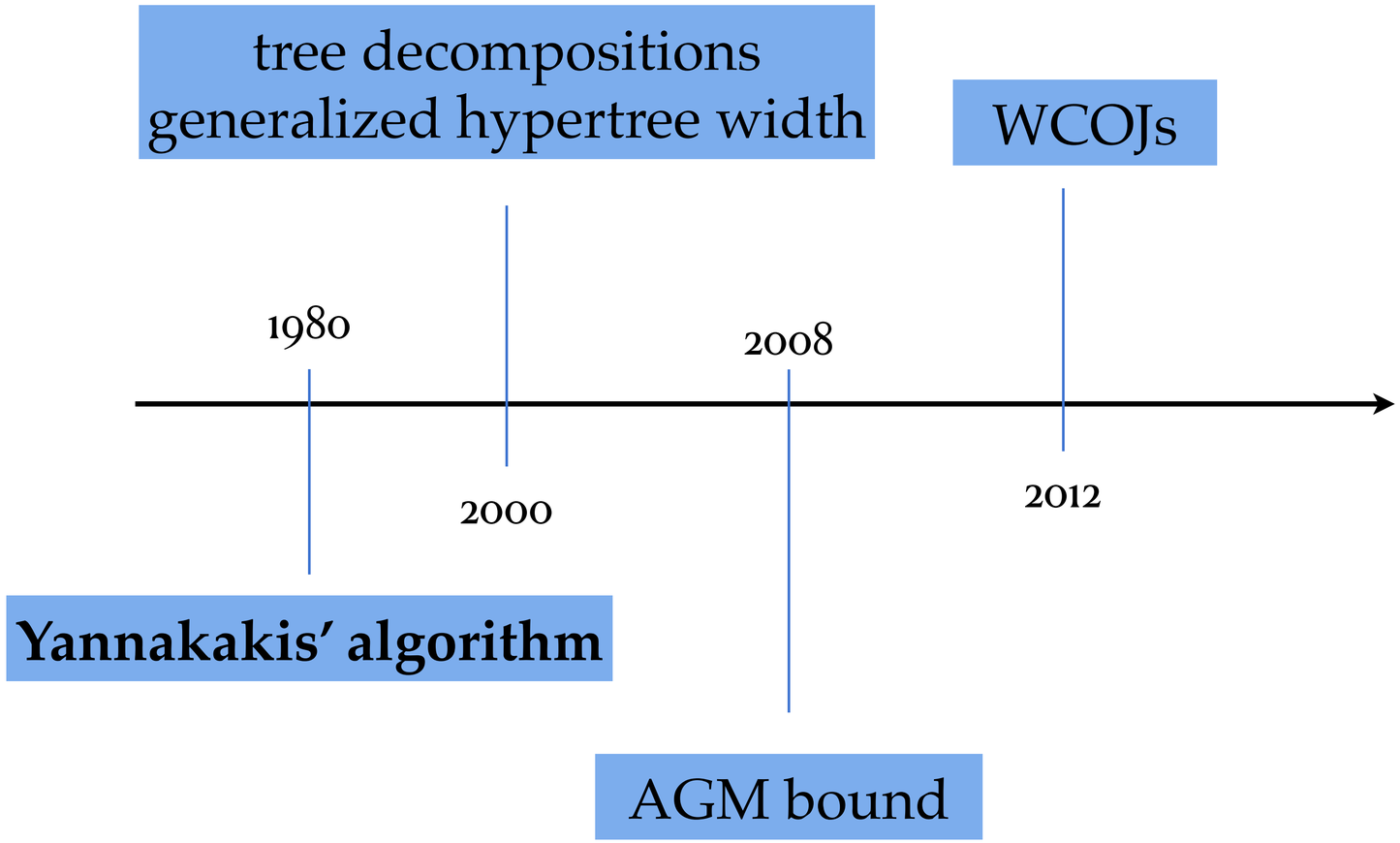
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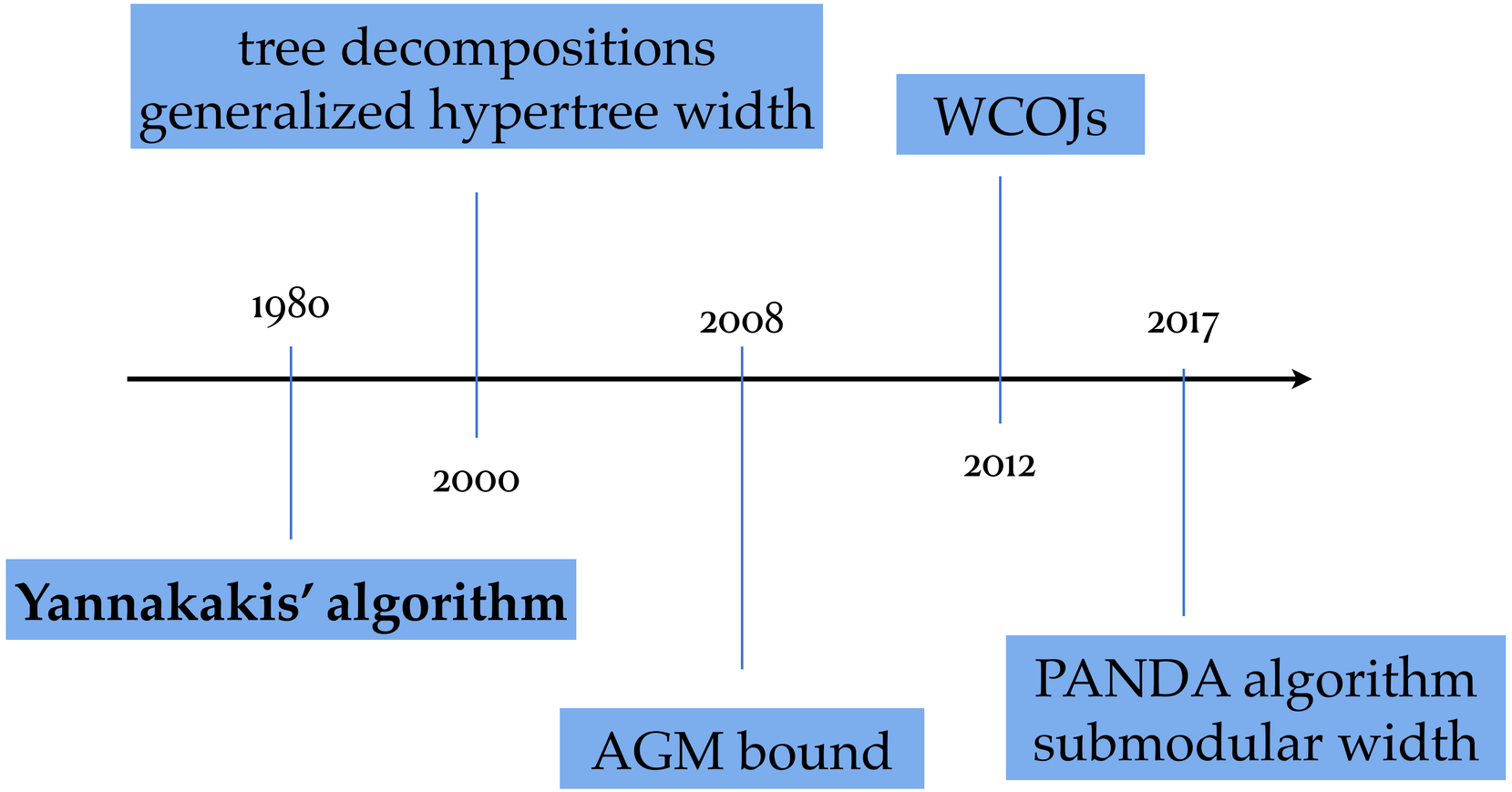
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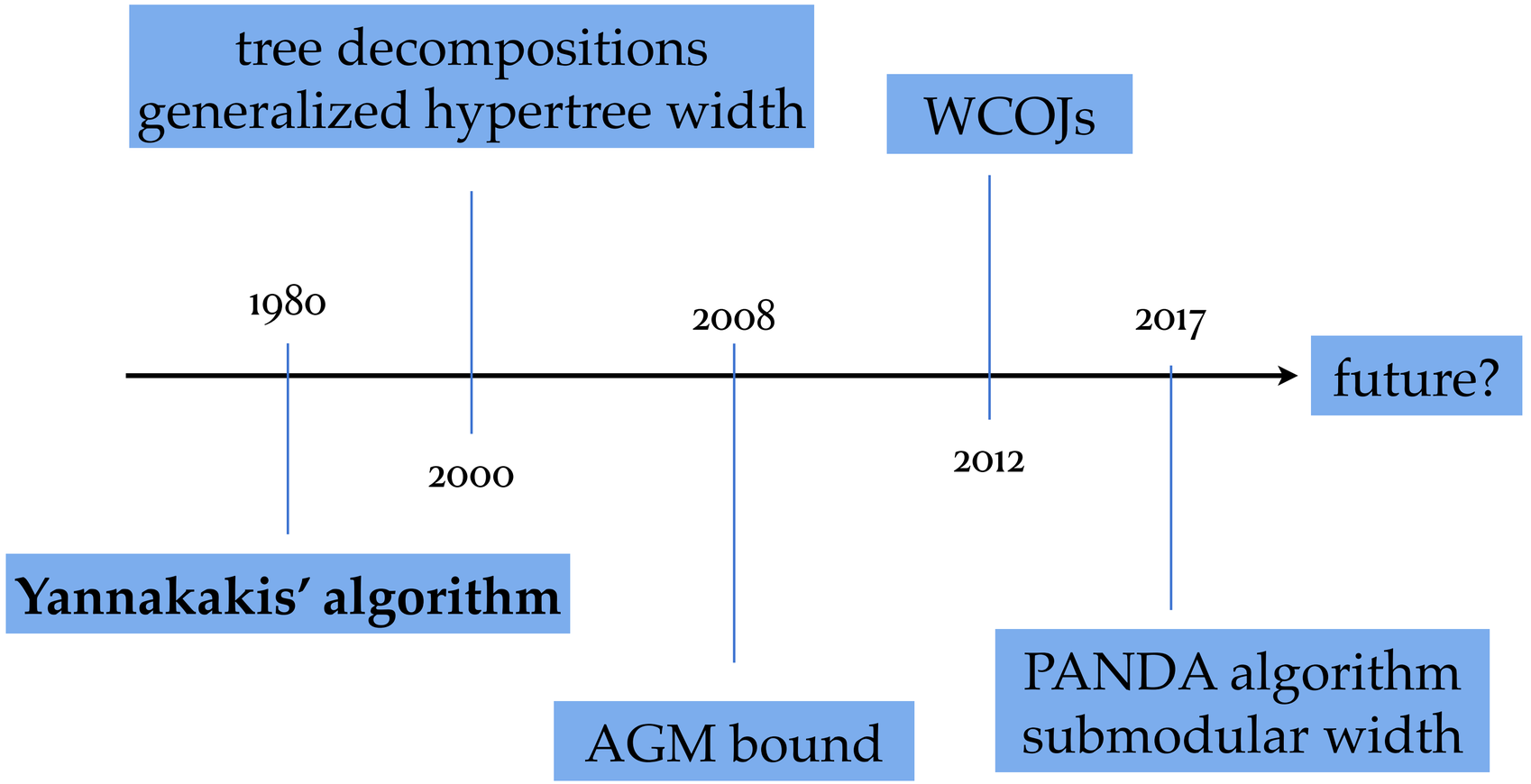
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future?
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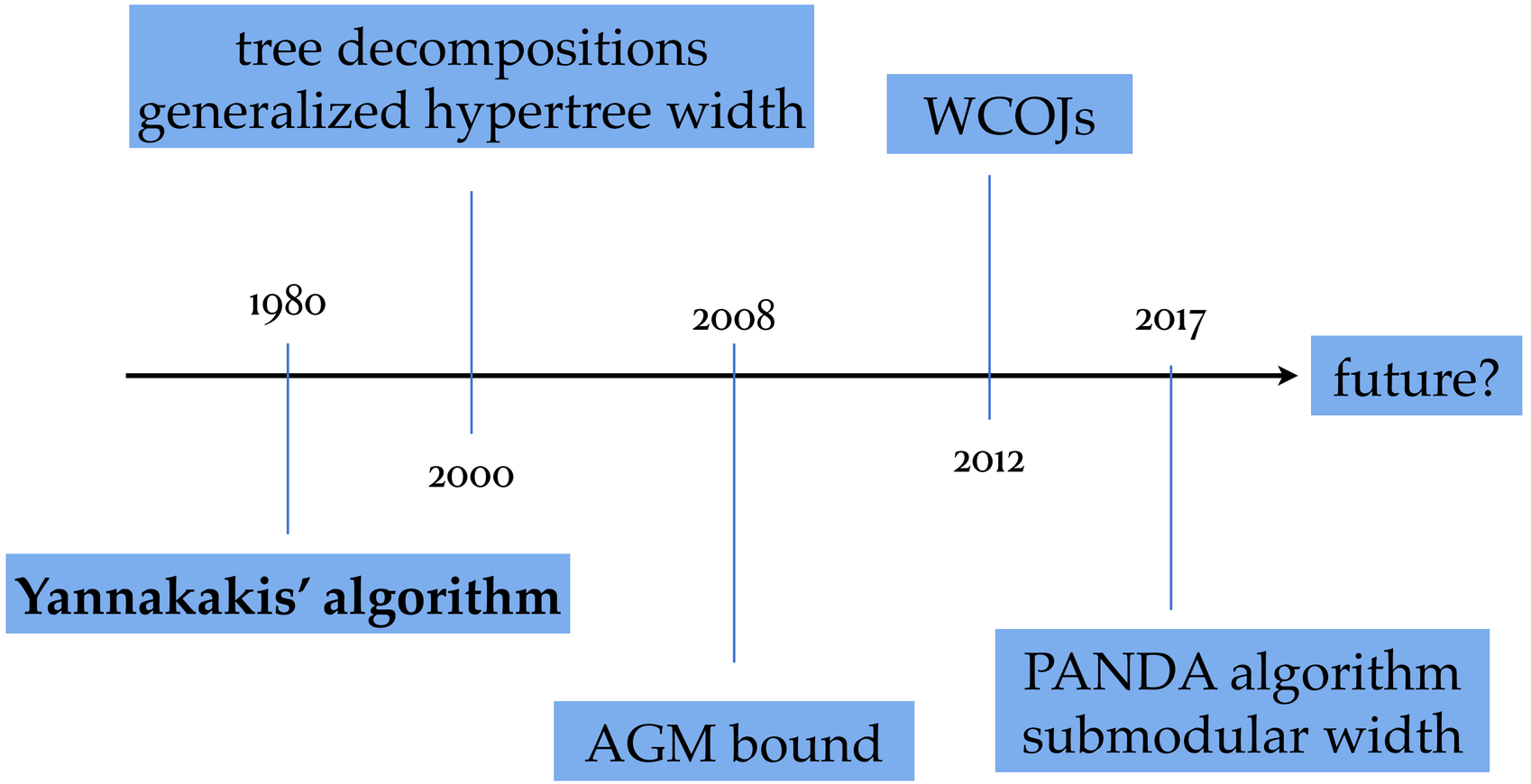


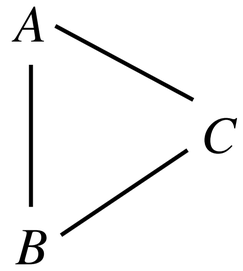
5 Key Ideas:

1. tree decompositions
2. data partitioning
3. using statistics
4. enumeration
5. the algebraic lens

Some background

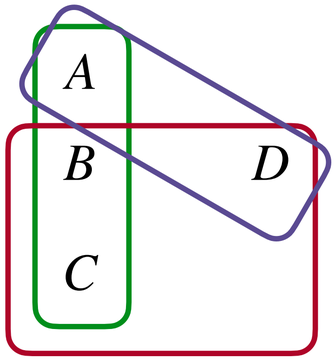
this talk: queries with **natural joins** + **projections**

$\pi_{\emptyset}(R(A,B) \bowtie S(B,C) \bowtie T(C,A))$



Boolean: project out everything

$R(A,B,C) \bowtie S(B,C,D) \bowtie T(A,D)$



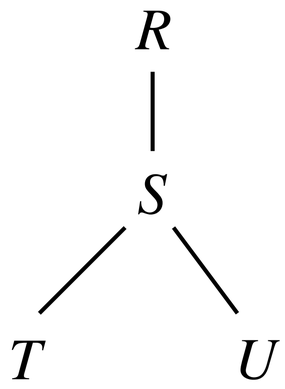
Full: no projections

Yannakakis' Algorithm

[Yannakakis - '81]

$$R(A, B) \bowtie S(B, C) \bowtie T(C, D) \bowtie U(C, E)$$

join tree



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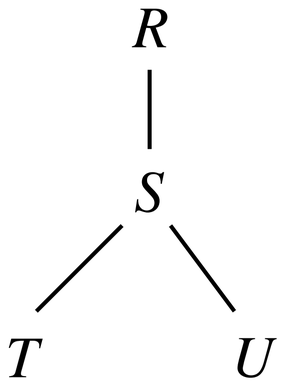
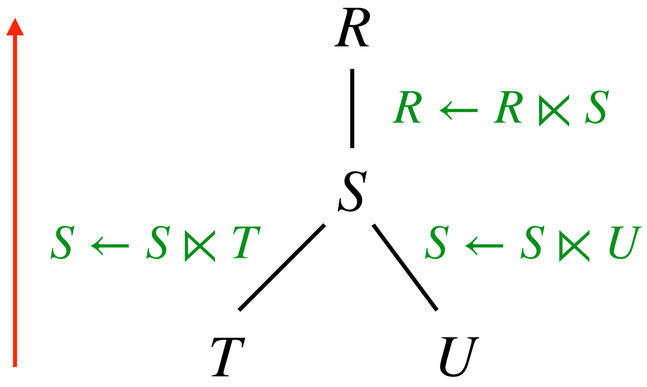


1 - semijoin phase



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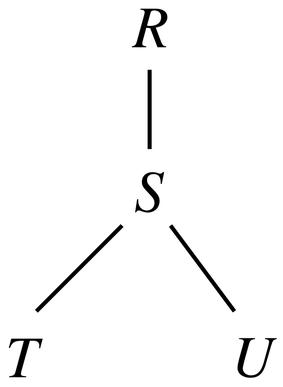

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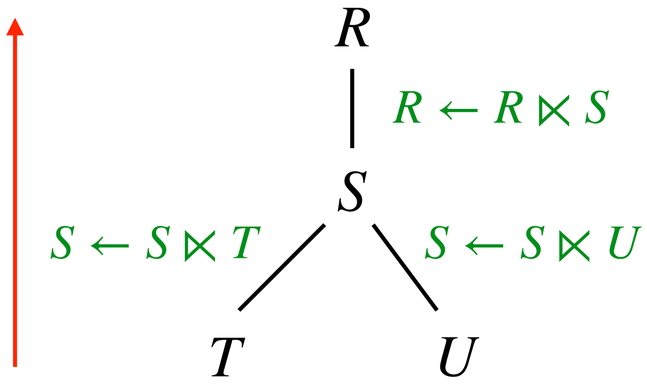
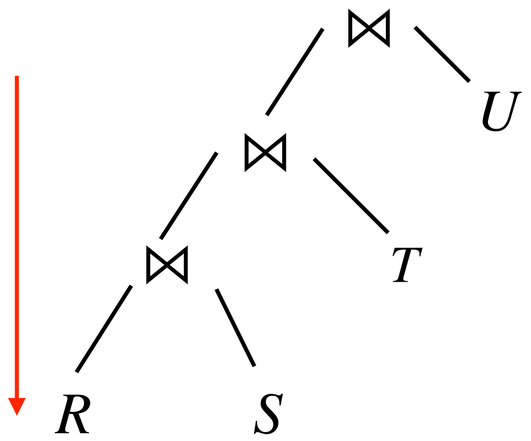


2 - join phase



Key property: any intermediate join size is bounded by the output size

Yannakakis' Algorithm

[Yannakakis - '81]

joins that have a join tree

Yannakakis' algorithm comes with runtime guarantees for **acyclic joins**

Full	Boolean	General
$O(N + OUT)$	$O(N)$	$O(N \cdot OUT + OUT)$

input output

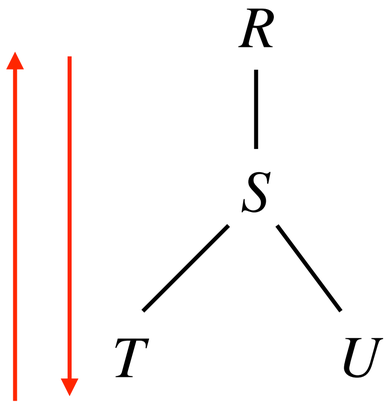
Predicate Transfer: Yannakakis in Practice

- Yannakakis can be slow because of the high cost of a semi-join
- We can replace the semi-joins by (approximate) Bloom filters [Yang et al. - '24]
- We obtain **speedup** + **robustness** to bad query plans

$$R(A,B) \bowtie S(B,C) \bowtie T(C,D) \bowtie U(C,E)$$

1 - transfer phase

forward-backward transfer of Bloom filters



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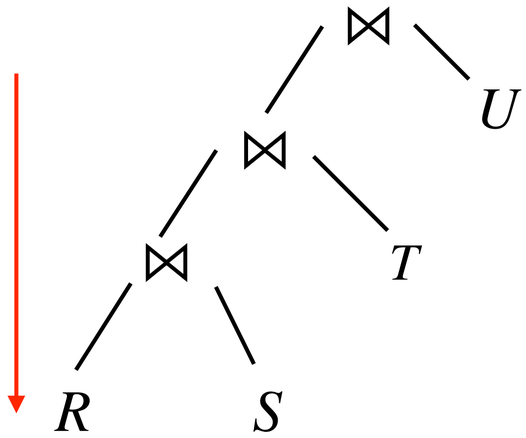

Predicate Transfer: Yannakakis in Practice

Robustness Factor := max execution time / min execution time

[Zhang et al. - **'25**]

worst join order **best join order**

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[Zhang et al. - '25]

worst join order **best join order**

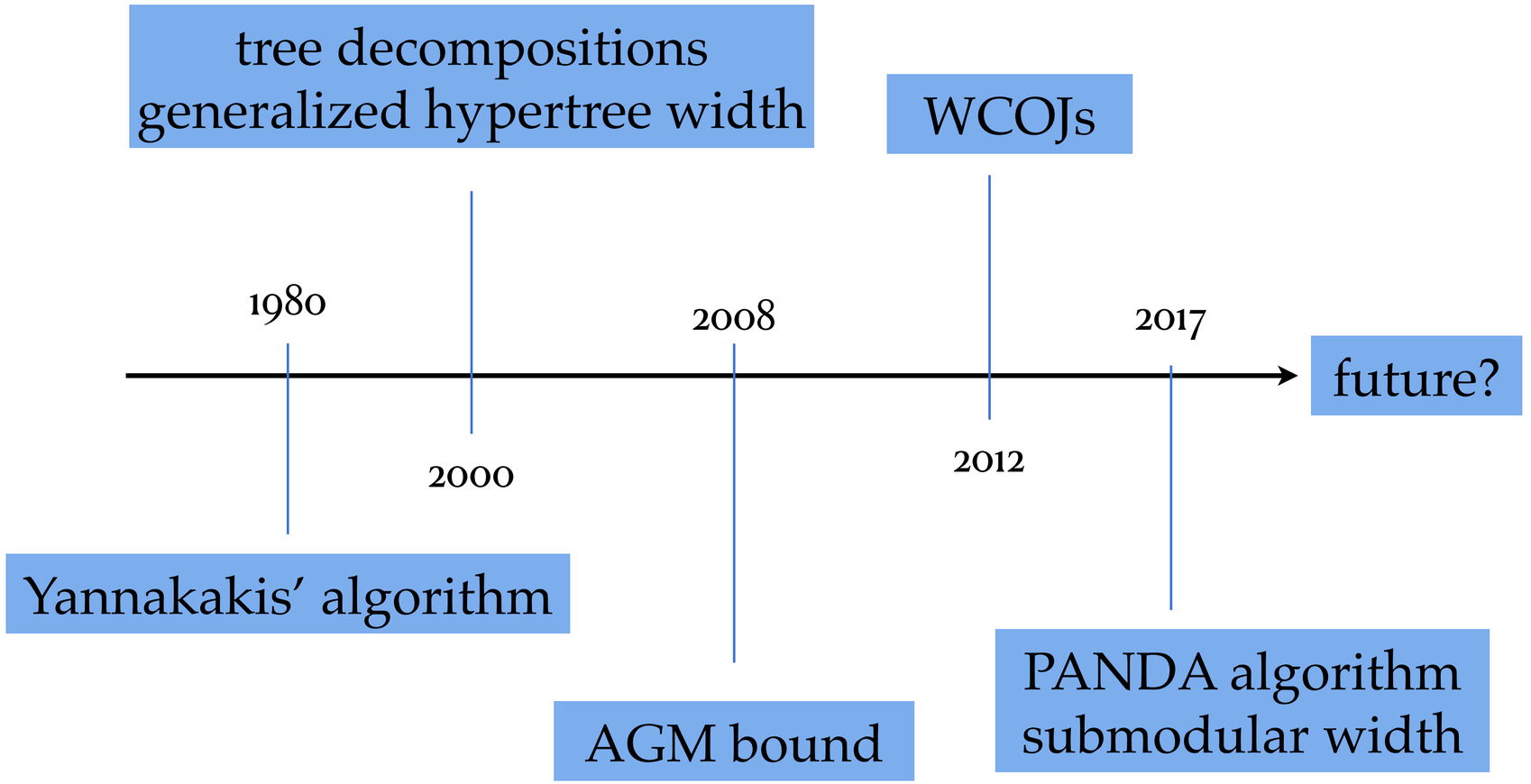
Table 1: Robustness Factors for left-deep joins.

RF	TPC-H			JOB			TPC-DS		
	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max
DuckDB	2.7	1.2	9.3	30.4	1.1	371	7.2	1.0	224
RPT	1.3	1.2	1.5	1.2	1.0	1.6	1.1	1.0	1.5

Table 2: Robustness Factors for bushy joins.

RF	TPC-H			JOB			TPC-DS		
	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max
DuckDB	5.1	1.2	13.7	120	1.1	1747	35.0	1.0	1226
RPT	1.8	1.2	3.0	1.6	1.1	7.7	1.8	1.0	4.2

Roadmap

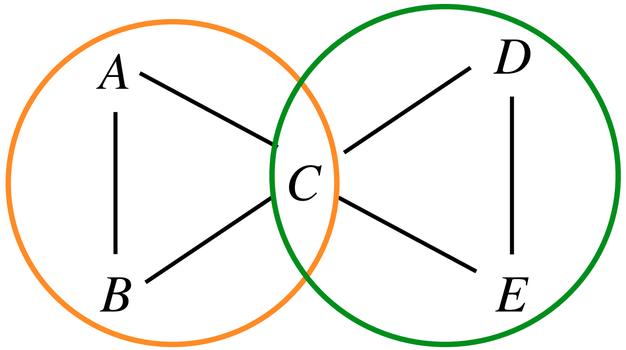


5 Key Ideas:

1. **tree decompositions**
2. data partitioning
3. using statistics
4. enumeration
5. the algebraic lens

1 - Tree Decompositions

- Acyclic joins have plans where the intermediate size is bounded by the output size.
- What can we do for cyclic joins?



$T_1(A, B, C) \bowtie T_2(C, D, E)$

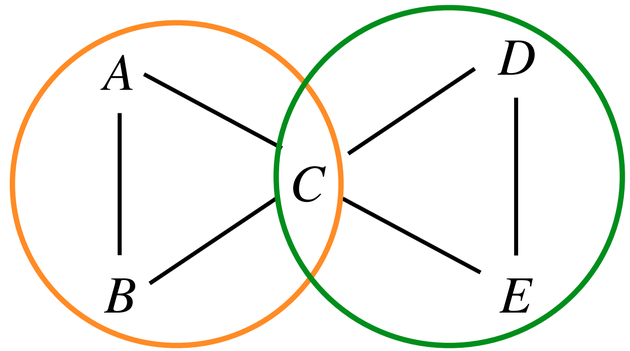
$O(N^2)$ $O(N^2)$

IDEA: make the join acyclic by computing larger intermediate relations

$O(N^2 + OUT)$

Tree Decompositions - Definition

- Like a join tree, but each node is an intermediate relation (**bag**)
- Every relation must be included in some bag



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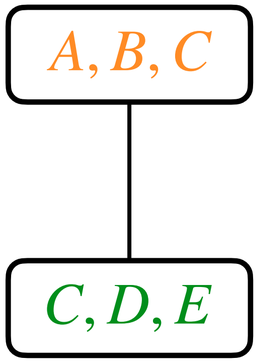
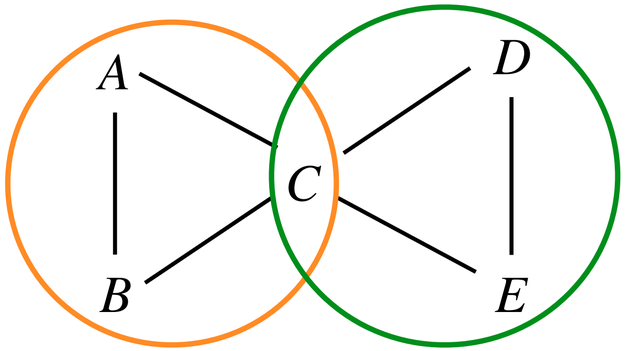
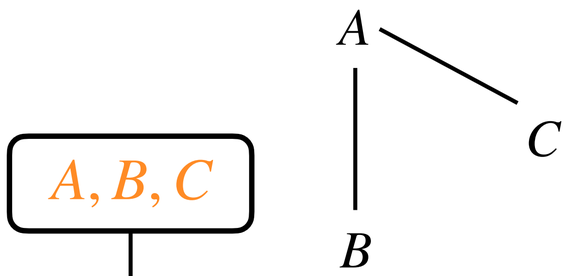


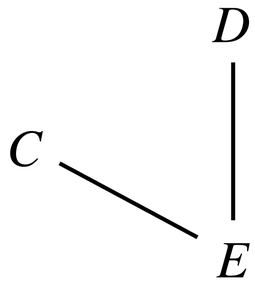
Generalized HyperTree Width



integral edge cover w: smallest subset of relations that "covers" all attributes in a bag



$w = 2$



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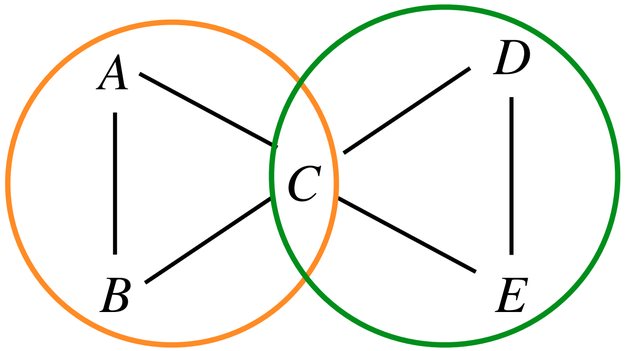

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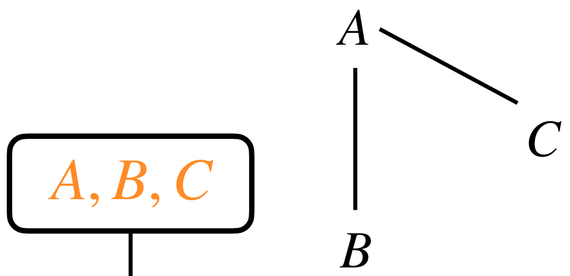

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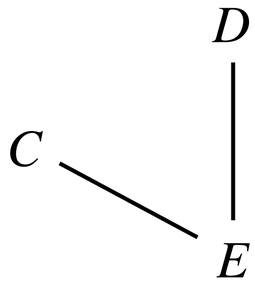


$w = 2$

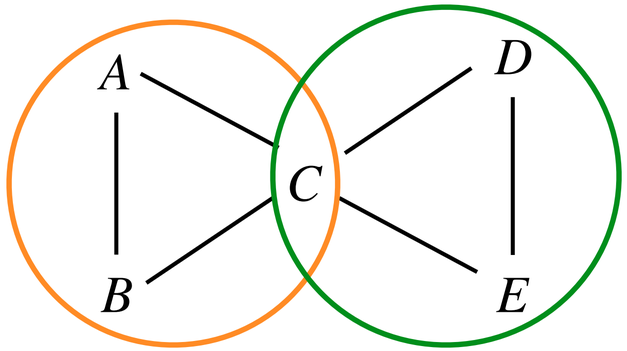
$$\text{ghw}(Q) := \min_{\text{decomp } T} \max_{\text{bag } B} w(X_B)$$

ghw = 2

Runtime Result

If Q is a full join, we can evaluate it in time

$$O\left(N^{ghw(Q)} + OUT\right)$$

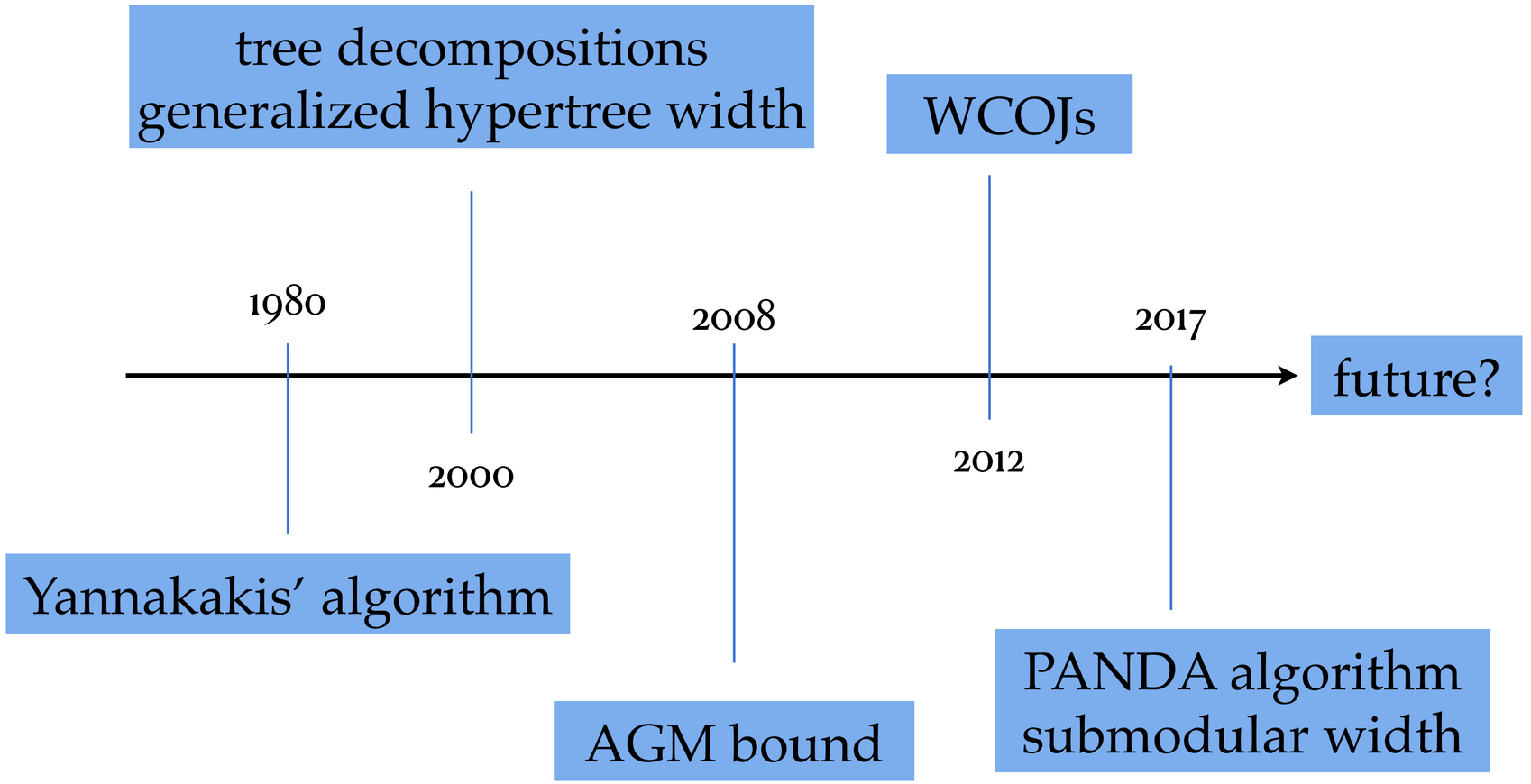


$T_1(A, B, C) \bowtie T_2(C, D, E)$

$O(N^2 + OUT)$

The algorithm can be expressed as a join-project plan!

Roadmap



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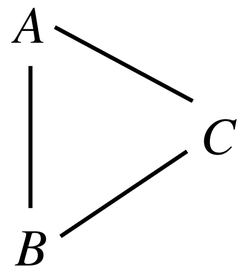
2 - Data Partitioning

Join-at-a-time algorithms are **suboptimal!**

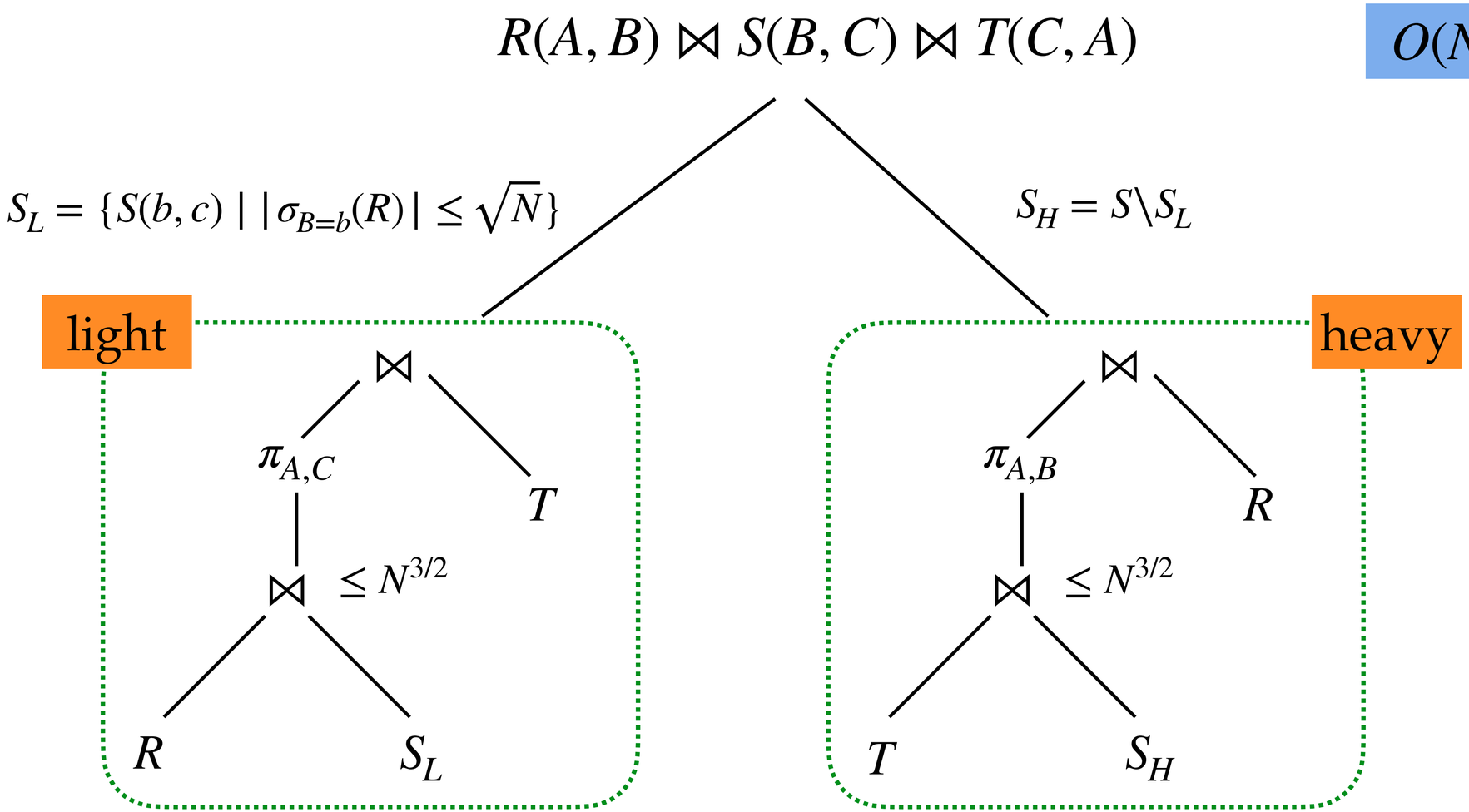


best binary join algorithm: $O(N^2)$

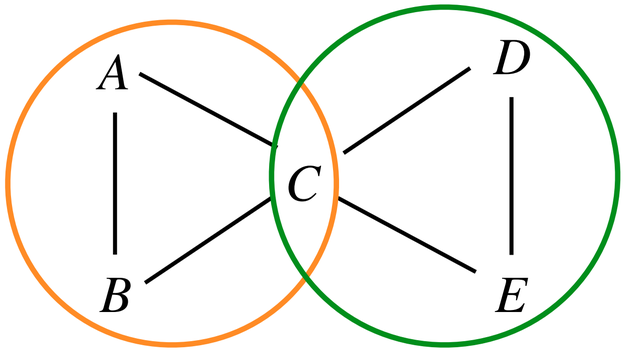
we can do better: $O(N^{3/2})$

We can do better by **partitioning** the input data and applying a different join plan to each partition

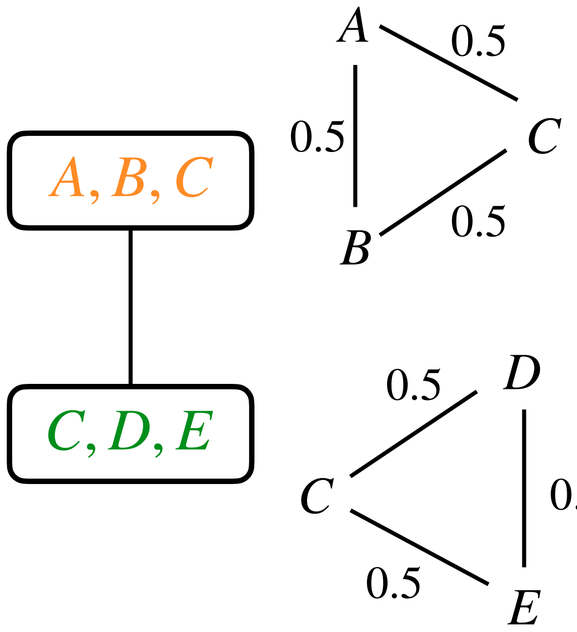
Data Partitioning: Example



Fractional Hypertree Width



fractional edge cover f: assign weights to relations to "cover" all attributes in a bag



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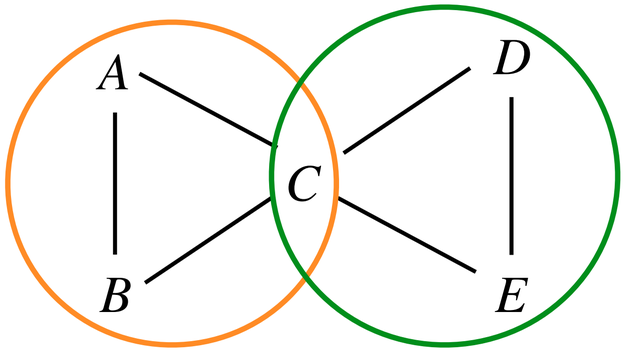

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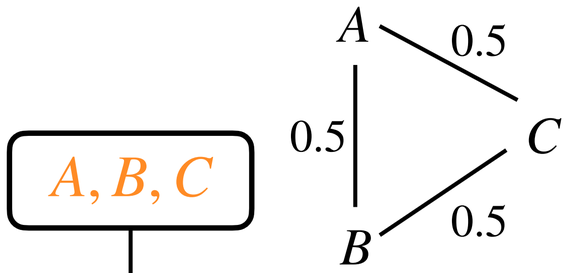

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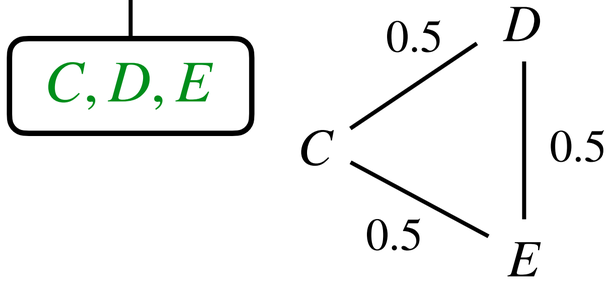

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Runtime Result Revisited

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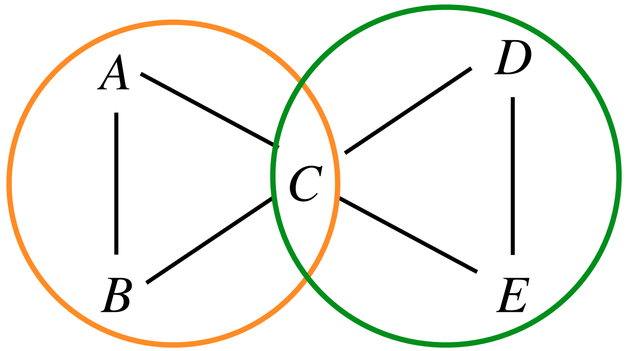
$$O\left(N^{fhw(Q)} + OUT\right)$$

[Ngo, Re, Rudra - '13]



$T_1(A, B, C) \bowtie T_2(C, D, E)$ ..... $O(N^{3/2} + OUT)$

The algorithm can be expressed as a join-project-**partition** plan!

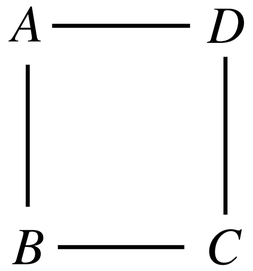
(Worst-case Optimal Joins)

- WCOJs have a running time matching the worst-case output size
- They often work in a attribute-at-a-time manner
  - Leapfrog Triejoin [Veldhuizen - '14]
  - GenericJoin [Ngo, Re, Rudra - '13]
  - FreeJoin [Wang, Willsey, Suciu - '23]

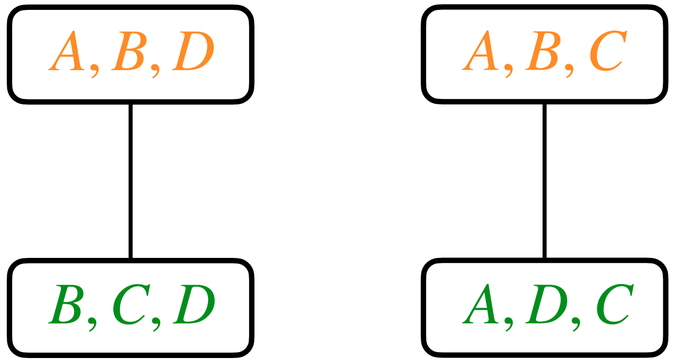
AGM bound [Atserias, Grohe, Marx - '08]

A lot of successful practical implementations!

Data Partitioning Again



We can partition data to use different tree decompositions!



fhw = 2

Runtime Result (Re)Revisited

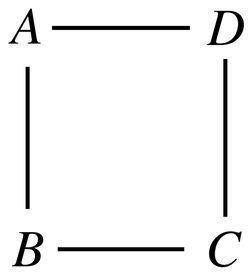
If Q is a full join, we can evaluate it in time

$$\tilde{O}\left(N^{subw(Q)} + OUT\right)$$

[Khamis, Ngo, Suciu - '17]

$$subw(Q) := \max_{\text{submodular } h} \min_{\text{decomp } T} \max_{\text{bag } B} h(X_B)$$

subw ≤ fhw ≤ ghw



$O(N^{3/2} + OUT)$

PANDA algorithm

Output-Sensitive Yannakakis

- When we have projections in an acyclic join, Yannakakis runs in time $O(N + N \cdot OUT)$
- We can design faster join algorithms by taking the output size OUT into account (using **data partitioning** again)

If Q is an acyclic join, we can evaluate it in time

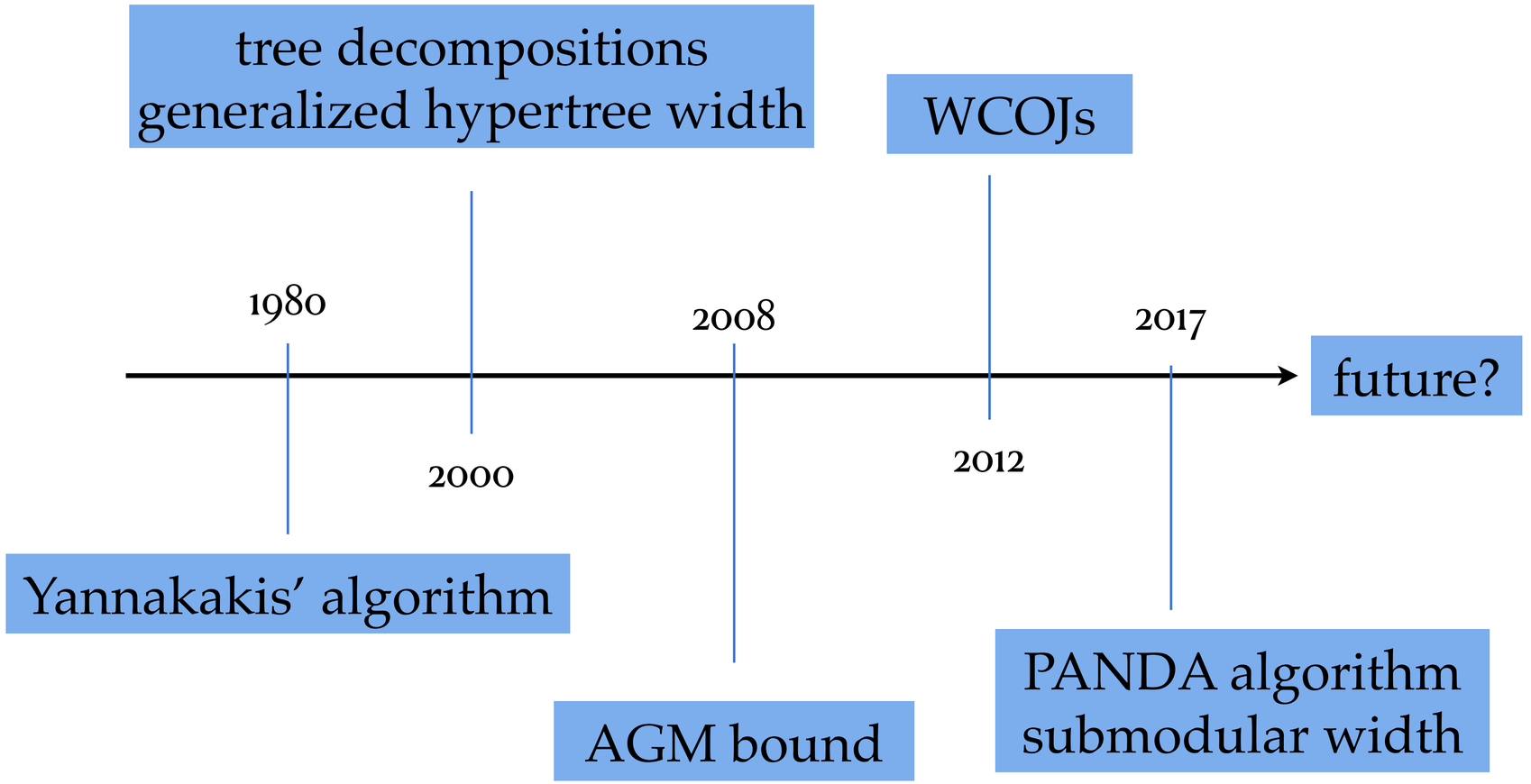
$$O(N + N \cdot OUT^{1-\epsilon})$$

[Deep, Zhao, Fan, Koutris - '**25**]

[Hu - '**25**]

$\pi_{A,C}(R(A,B) \bowtie T(B,C))$ ..... $O(N + N \cdot OUT^{1/2})$

Roadmap

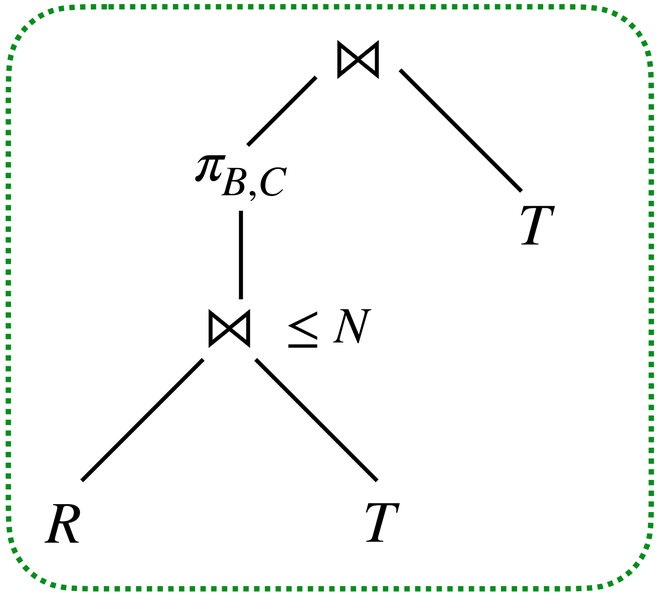


5 Key Ideas:

1. tree decompositions
2. data partitioning
3. **using statistics**
4. enumeration
5. the algebraic lens

3 - Using Statistical Information

By using more **statistical information**, we can design faster join algorithms



primary key

$R(A, B) \bowtie S(B, C) \bowtie T(C, A)$

runs in time only $O(N)$

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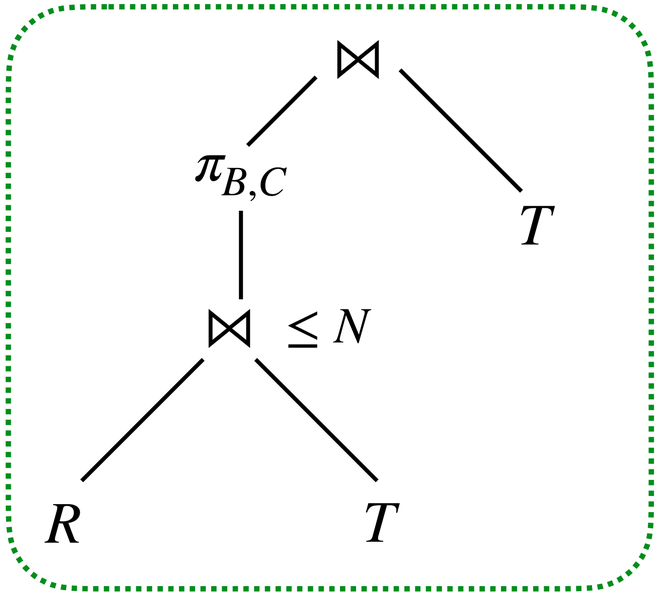


primary key

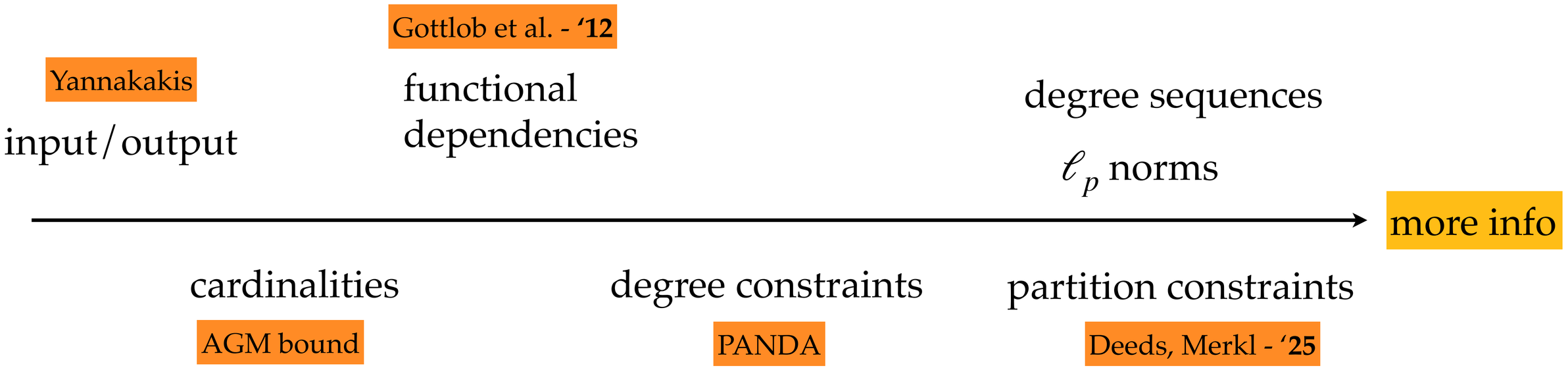
$R(A, B) \bowtie S(B, C) \bowtie T(C, A)$

runs in time only $O(N)$

Using more detailed statistics

- restricts what is the worst-case behaved input
- closes the gap between theory and practice

3 - Using Statistical Information



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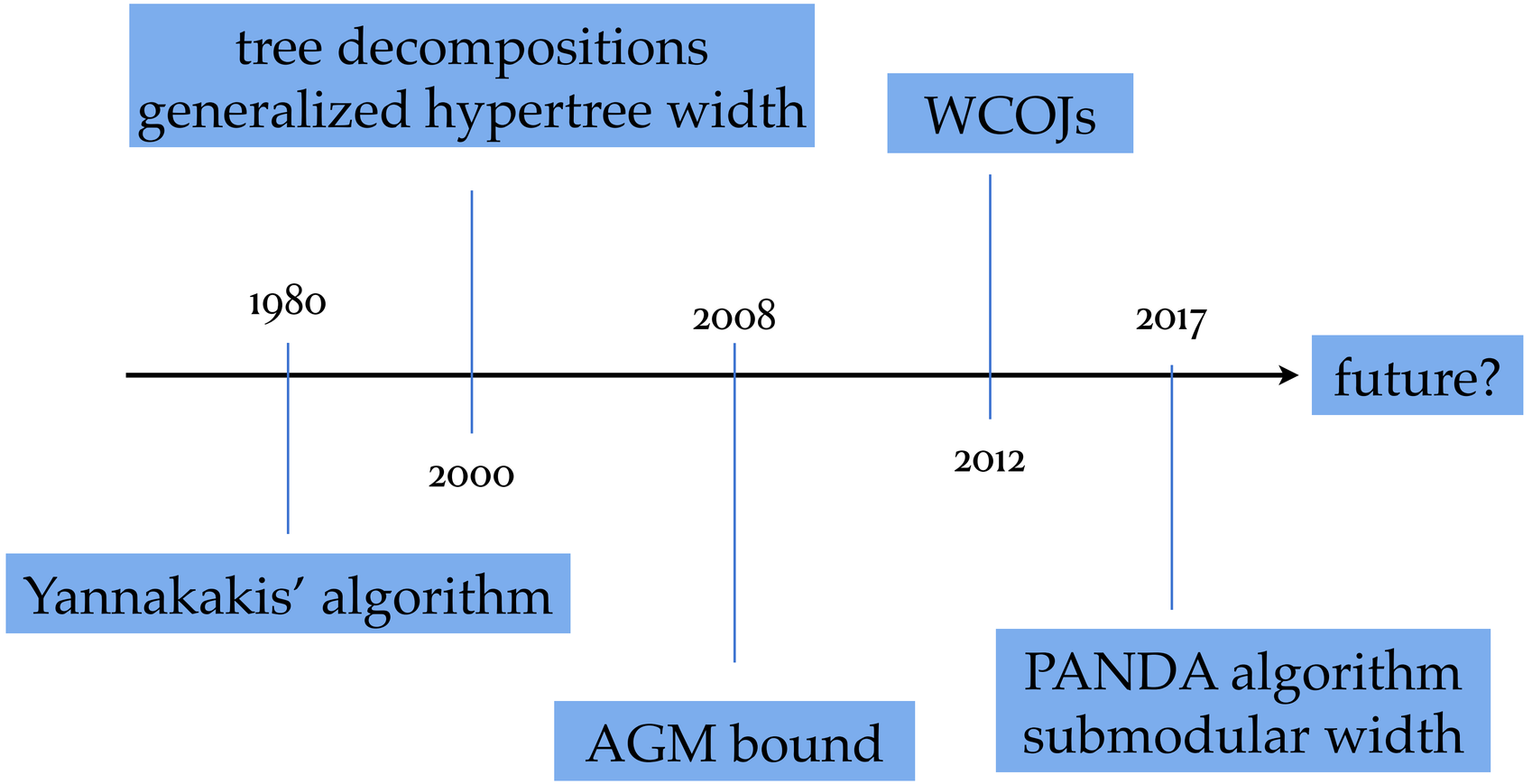

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4 - Enumeration

We can design join algorithms with stronger **enumeration** guarantees

- useful for LIMIT queries
- constant delay enumeration [Bagan et al. - '07]

$O(N + OUT)$ vs After $O(N)$ preprocessing time, we can enumerate the output while spending only constant time between two consecutive outputs

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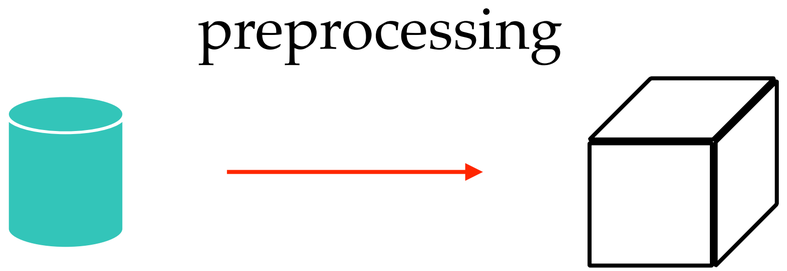

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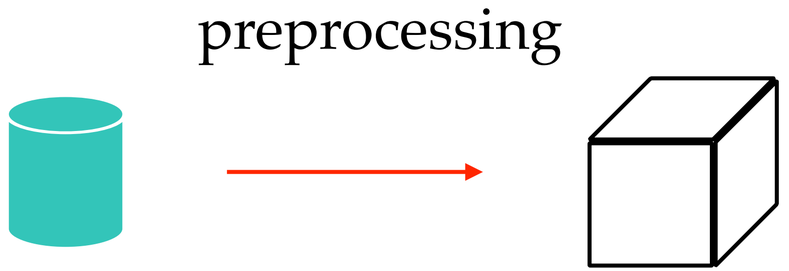

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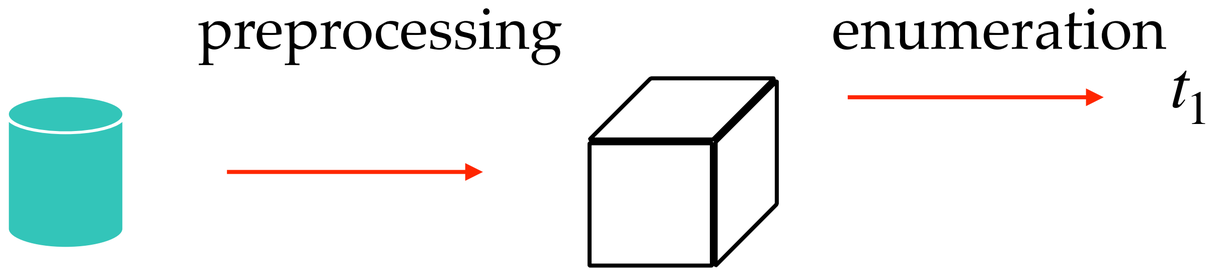

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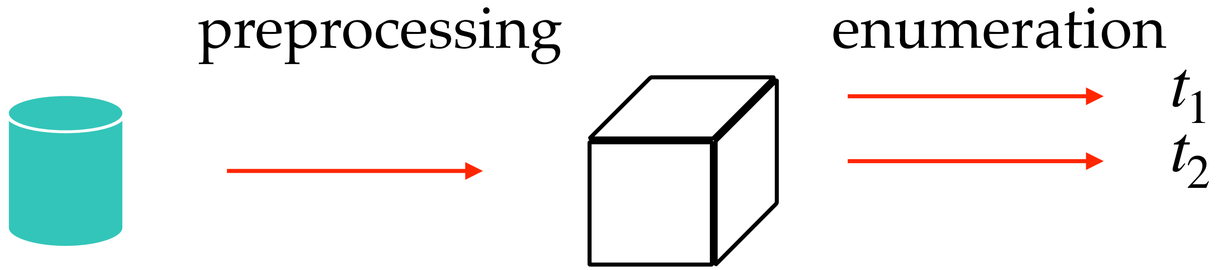

$O(N + OUT)$ vs

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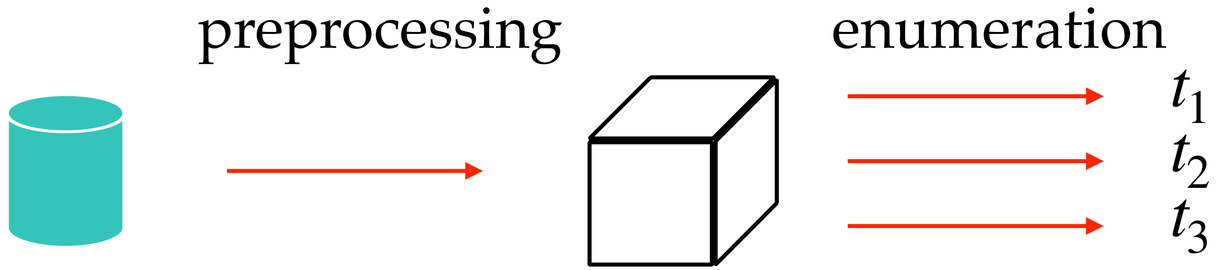


$O(N + OUT)$ vs

After $O(N)$ preprocessing time, we can enumerate the output while spending only constant time between two consecutive outputs

Runtime Result Revisited

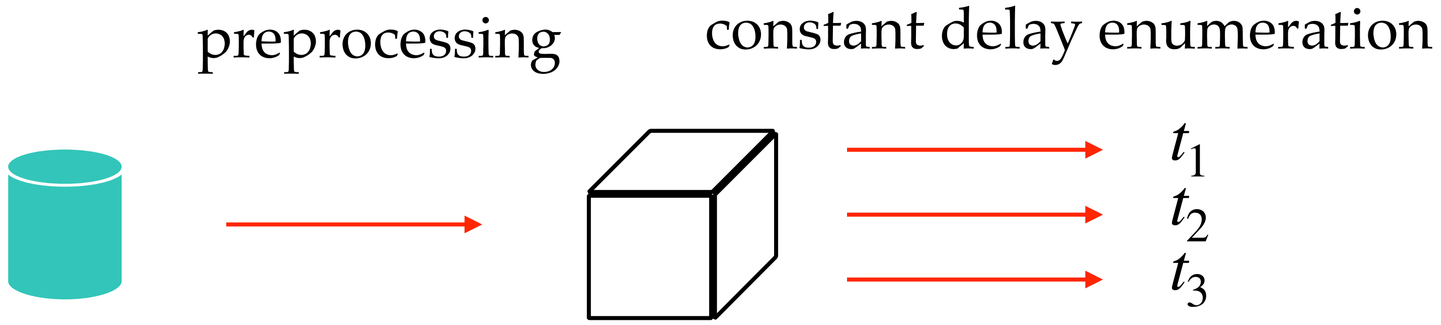
If Q is a full join, we can do constant delay enumeration with preprocessing time

$$\tilde{O}\left(N^{subw(Q)}\right)$$

Acyclic queries only need linear preprocessing time!

Factorized Databases

[Olteanu,Zavodny - '16]



The intermediate data structure can be viewed as a compressed (*factorized*) representation of the join output that can be decompressed efficiently

Joins with Ranked Order

1. Joins where we only want top-ranked results (ORDER BY + LIMIT)
2. We can construct algorithms with strong enumeration guarantees
3. Works well in practice - it avoids intermediate *expensive materializations*

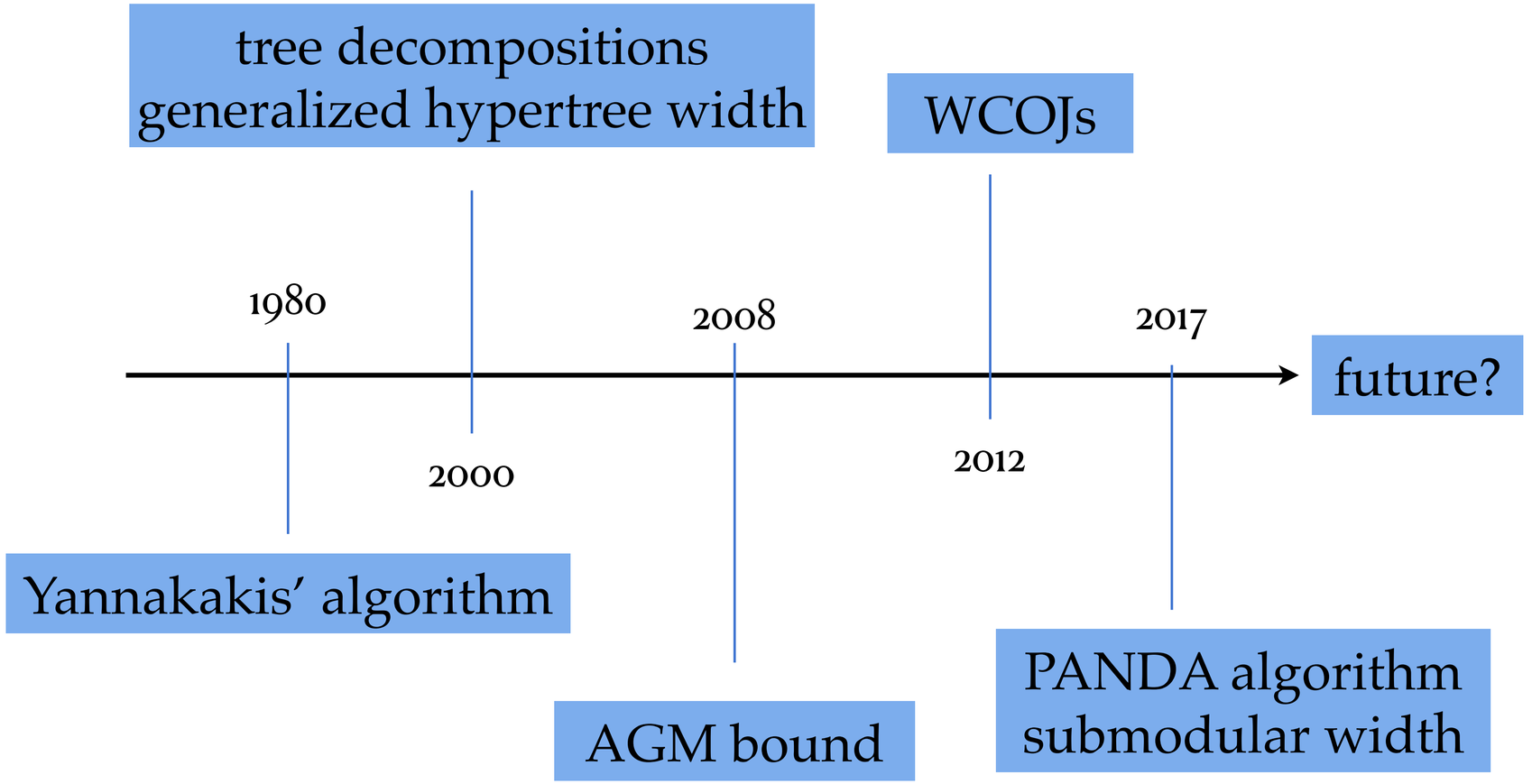
```
SELECT *
FROM R, S
WHERE R.A = S.B
ORDER BY S.C + R.D
LIMIT 10
```

For acyclic joins, after $O(N)$ preprocessing time, we can enumerate the output while spending only **logarithmic** time between two consecutive outputs

[Deep, Koutris - '**21**]

[Tziavelis et al. - '**20**]

Roadmap



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5. **the algebraic lens**

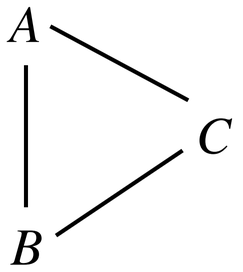
5 - The Algebraic Lens

Most existing join algorithms can be *lifted* to work on general algebraic structures called **semirings** [Green, Karvounarakis, Tannen - '**21**]

- aggregation becomes simple - sum, count, min, max
- extremely useful for ML pipelines (FAQs, factorized databases)

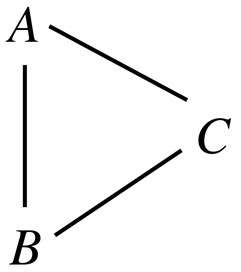
- join: **multiplication** ($\times$)
- project, union: **addition** ($+$)

Triangles with Semirings



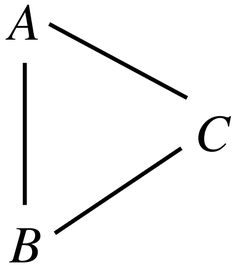
$(\{0,1\}, \vee , \wedge ,0,1)$

\+ **Boolean semiring** = is there is a triangle?



$(\mathbb{N}, \min , + ,\infty,0)$

\+ **tropical semiring** = minimum weight triangle



$(\mathbb{N}, + , \cdot ,0,1)$

\+ **arithmetic semiring** = how many triangles?

The Algebraic Lens

If Q is an acyclic full join, we can compute it over any semiring it in time

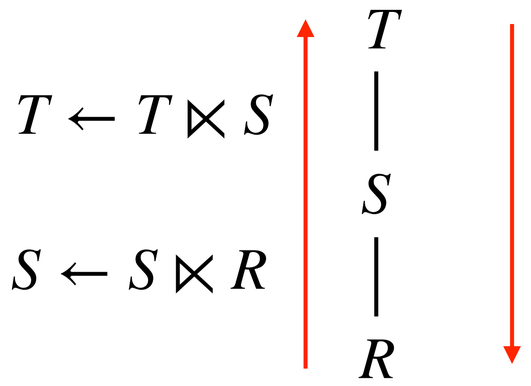
$$O(N + OUT)$$

$$R(A, B) \bowtie S(B, C) \bowtie T(C, D)$$

and many other results…

FAQs [Khamis, Ngo, Rudra - '**16**]

[Khamis et al. - '**19**]



1. semijoin

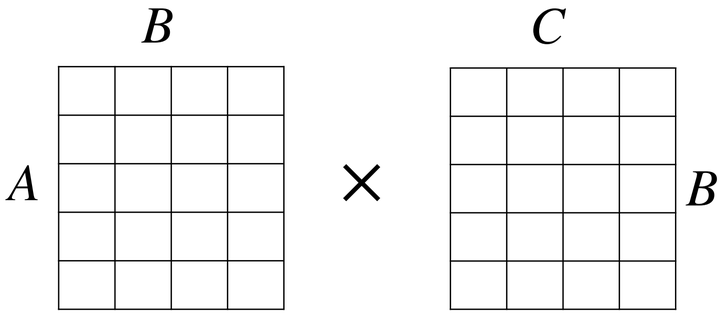
2. join + (partial) aggregation

What comes next?

Towards Even Faster Joins

We can join faster using **Fast Matrix Multiplication (FMM)**

$$\pi_{A,C}(R(A,B) \bowtie S(B,C))$$



- combinatorial: $O(n^3)$
- FMM: $O(n^\omega)$, $\omega = 2.371339$

$$\pi_{\emptyset}(R(A,B) \bowtie S(B,C) \bowtie T(C,A))$$

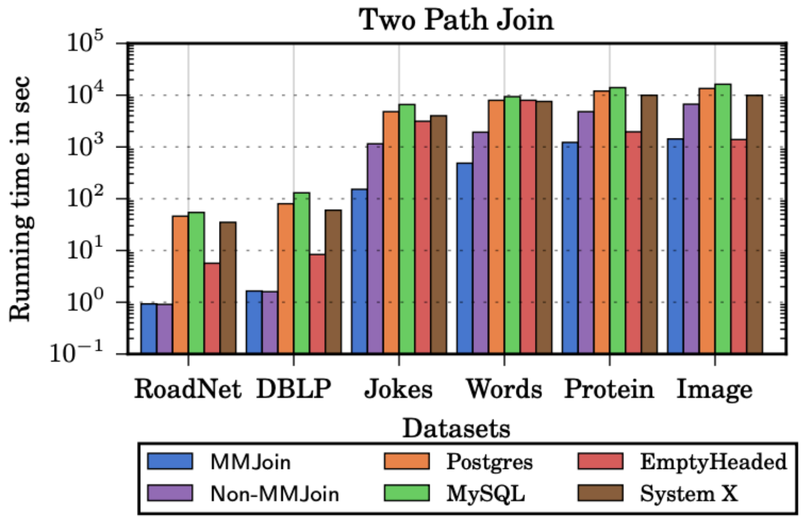
combinatorial: $O(N^{3/2})$
with FMM: $O(N^{\frac{2\omega}{1+\omega}})$

[Alon, Yuster, Zwick - '97]

Towards Even Faster Joins

- FMM can be incorporated as a new operator in a query plan [Khamis, Hu, Suciu - '25]
- Using MM can also help in practice

[Deep, Hu, Koutris - '22]



Q1: Are there other ways to use FMM in query plans?

Q2: What other non-combinatorial techniques can we use?

Joins for any Semiring - how do we count?

- The PANDA algorithm *cannot count*!
- The key problem is the overlap of output tuples in tree decompositions
- We can count the join size in time $O(N^{\#\text{subw}})$ [Khamis et al. - '**19**]

$$\text{subw} \leq \#\text{subw} \leq \text{fhw}$$

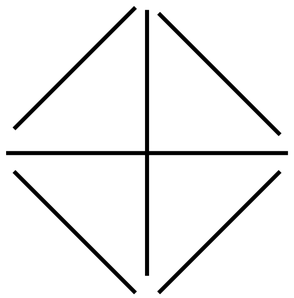
Q3: What is the best algorithm that counts the size of a join?

Lower Bounds for Joins

- To show that join algorithms are optimal, we need lower bounds
- Not (currently) feasible to show unconditional bounds
- We use *conditional lower bounds* by reducing from problems where we believe we have an optimal algorithm

k-clique problem on n vertices

$\Omega(n^k)$



(conditional) lower bounds for all joins

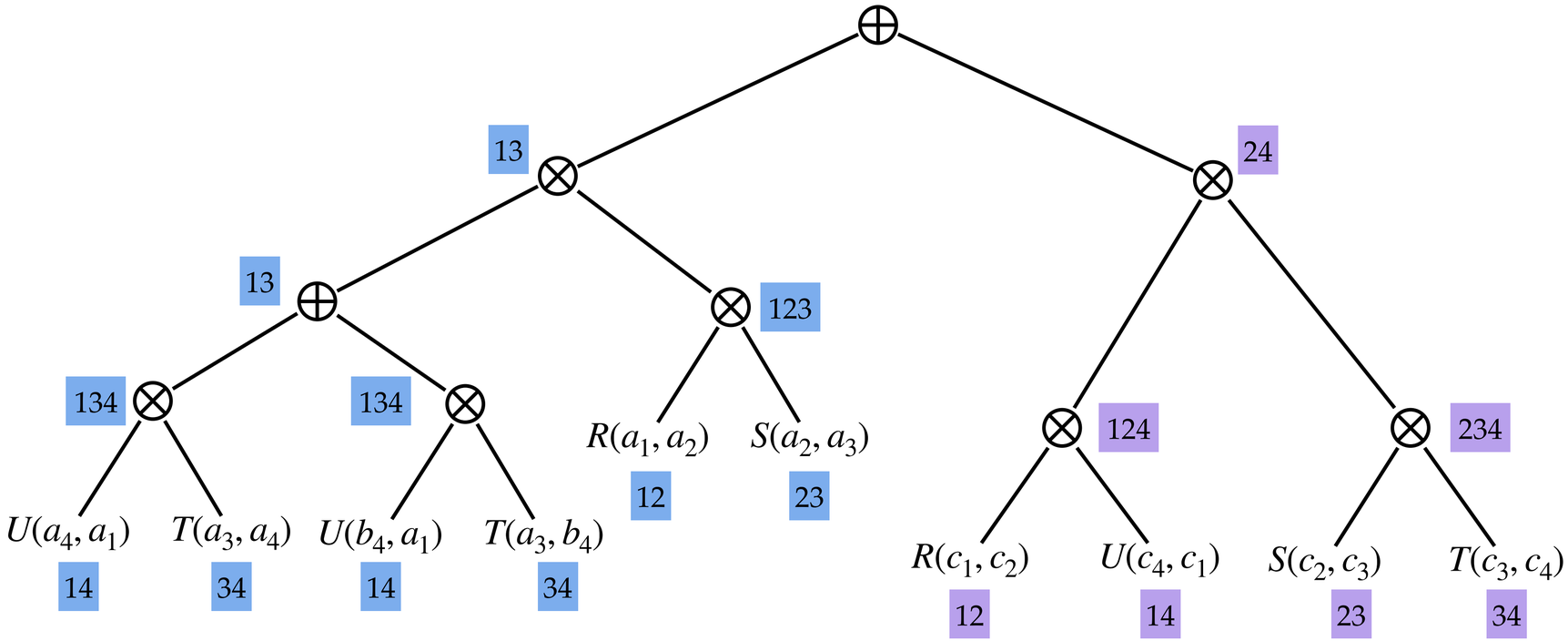
$\Omega(N^{clemb})$ [Fan, Koutris, Zhao - '23]

clemb ≤ subw ≤ fhw

Q4: What other conditional lower bounds should we use?

Lower Bounds for Joins

- Another approach is to restrict the algorithm
- In particular, we ask that the algorithm structure is encoded via a **circuit**
- We can show matching upper + lower bounds *for any join* [Fan, Koutris, Zhao - '**24**]

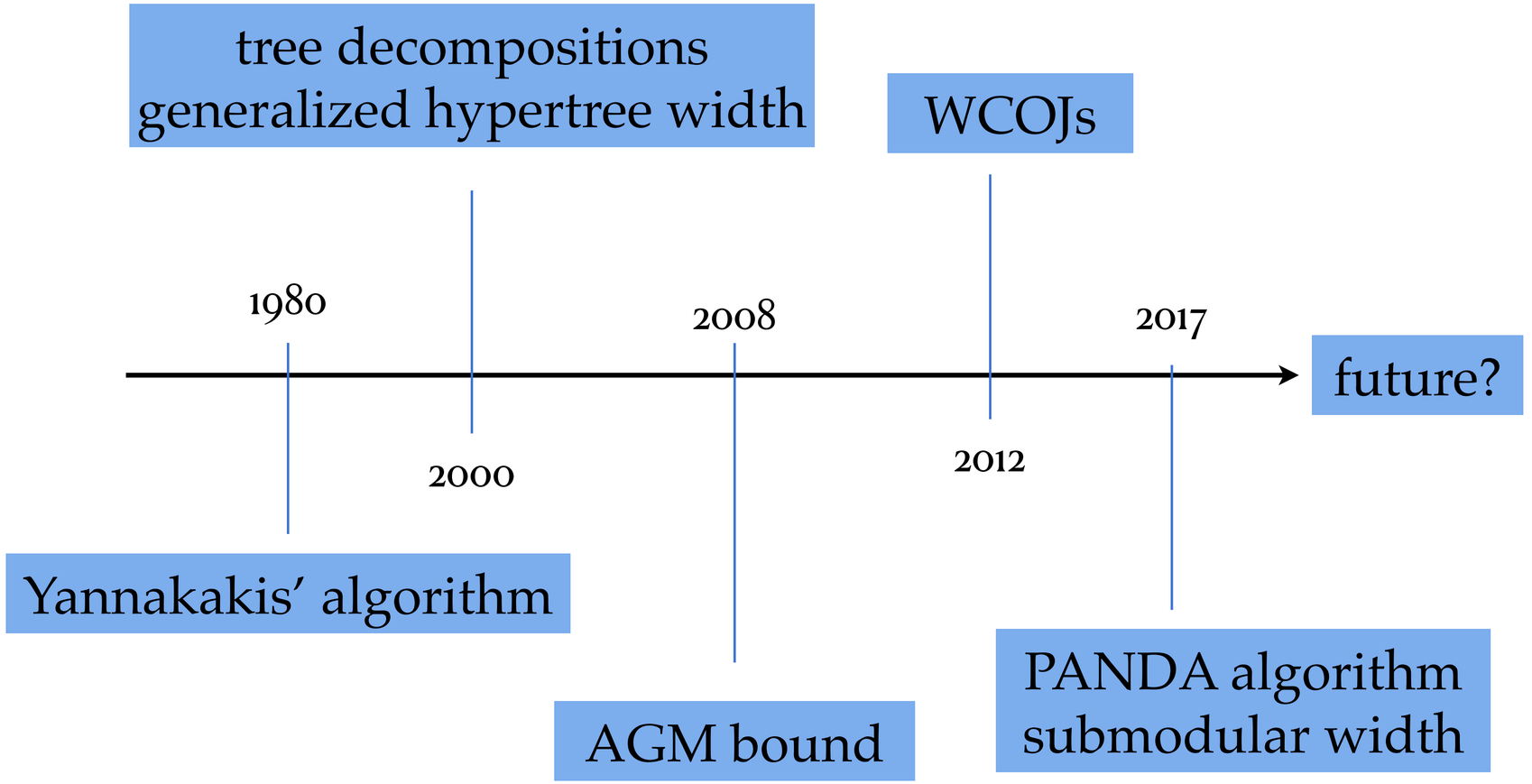


From Theory to Practice

Q5: Which theoretical developments can be effective in practice?

- some have already been successful: WCOJs, predicate transfer
- for some it is still unclear: enumeration?
- … and others are only theoretical (and will remain)

Thank You!



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