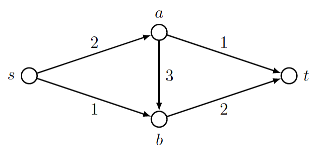
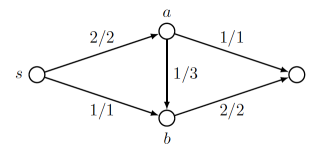
Network



Definition
A network N consists of:

- digraph (V, E)
- edge capacities $c : E \to [0, \infty)$
- source $s \in V$, which has indegree 0, and
- sink $t \in V$, which has outdegree 0.

Flow

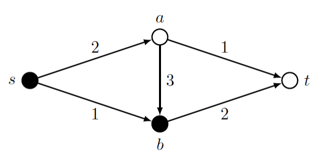


Definition
A flow is a mapping $f : E \to [0, \infty)$ satisfying:

- [capacity constraints] $(\forall e \in E)\, f(e) \le c(e)$
- [conservation constraints] $(\forall v \in V \setminus \{s,t\})\, f_{\text{in}}(v) = f_{\text{out}}(v)$
  where $f_{\text{in}}(v) \doteq \sum_{u \in V : e \doteq (u,v) \in E} f(e)$ and
  $f_{\text{out}}(v) \doteq \sum_{u \in V : e \doteq (v,u) \in E} f(e)$.

Value of a flow: $\nu(f) \doteq f_{\text{out}}(s)$

Cut

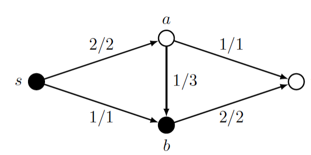


Definition
An st-cut is a partition (S, T) of V such that $s \in S$ and $t \in T$.

Note

- (S, T) is a partition of V if $S \cup T = V$ and $S \cap T = \emptyset$.
- (S, T) can alternately be written as $(S, \overline{S})$ where $\overline{S} \doteq V \setminus S$.
- Term "cut" is sometimes also used to denote the set of edges that cross the cut, i.e., $(E \cap S \times T) \cup (E \cap T \times S)$.

Invariance Property



Statement
For every flow f and st-cut (S, T) in a network $N = (V, E, c, s, t)$

$$f_{\text{out}}(S) - f_{\text{in}}(S) = \nu(f),$$

where $f_{\text{out}}(S) \doteq \sum_{e \in E \cap S \times T} f(e)$ and $f_{\text{in}}(S) \doteq \sum_{e \in E \cap T \times S} f(e)$.

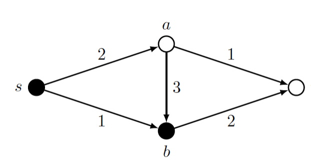
Proof of Invariance Property

$$\begin{array}{rclcll}
\nu(f) & \doteq & f_{\text{out}}(s) & - & f_{\text{in}}(s) & \\
0 & = & f_{\text{out}}(v) & - & f_{\text{in}}(v) & v \in S \setminus \{s\} \\
\hline
\nu(f) & = & \sum_{v \in S} f_{\text{out}}(v) & - & \sum_{v \in S} f_{\text{in}}(v) & \\
 & = & \sum_{v \in S} \sum_{e \doteq (v,u) \in E} f(e) & - & \sum_{v \in S} \sum_{e \doteq (u,v) \in E} f(e) & \\
 & \overset{(*)}{=} & \sum_{e^* \in E} c(e^*) f(e^*) & & & \\
 & = & \sum_{e \in E \cap S \times T} f(e) & - & \sum_{e \in E \cap T \times S} f(e) & \\
 & = & f_{\text{out}}(S) & - & f_{\text{in}}(S) &
\end{array}$$

(*)

type of $e^* \in E$	$c(e^*)$
$S \times T$	1 - 0 = 1
$S \times S$	1 - 1 = 0
$T \times S$	0 - 1 = -1
$T \times T$	0 - 0 = 0

Cut



Definition
An st-cut is a partition (S, T) of V such that $s \in S$ and $t \in T$.

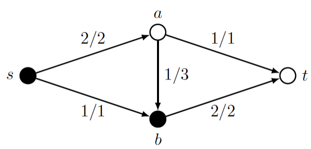
Capacity of an st-cut
$c(S, T) \doteq \sum_{e \in E \cap S \times T} c(e)$

Weak duality

Statement

For every flow f and st-cut (S, T) in a network $N = (V, E, c, s, t)$

$$\nu(f) \leq c(S, T).$$



Proof of Weak duality

$$\begin{aligned} \nu(f) &= \sum_{e \in E \cap S \times T} f(e) - \sum_{e \in E \cap T \times S} f(e) \\ &\leq \sum_{e \in E \cap S \times T} c(e) - \sum_{e \in E \cap T \times S} 0 \\ &\doteq c(S, T) \end{aligned}$$

Note

Equality $\nu(f) = c(S, T)$ holds iff

- $(\forall e \in E \cap S \times T)\, f(e) = c(e)$
- $(\forall e \in E \cap T \times S)\, f(e) = 0$.

Max Flow and Min Cut

Max flow

Input: network $N = (V, E, c, s, t)$
Output: flow f such that $\nu(f)$ is maximized

Min cut

Input: network $N = (V, E, c, s, t)$
Output: st-cut (S, T) such that $c(S, T)$ is minimized

Weak duality

$$\max_{\text{flow } f} \nu(f) \leq \min_{st\text{-cut } (S,T)} c(S, T)$$

We'll show next lecture that equality always holds (strong duality).

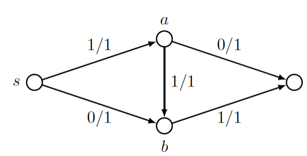
Path Augmentation - first attempt

Algorithm

1. Start with $f \equiv 0$.
2. While there is an st-path P along which more flow can be pushed, additionally push as much flow along P as possible.
3. Return f.

Issue

Bad prior choices may block further progress.



Residual Network

Consider a flow f in $N = (V, E, c, s, t)$.

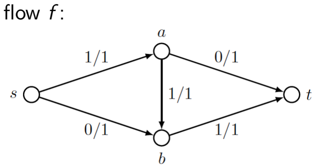
Definition

The residual network $N_f = (V, E_f, c_f, s, t)$ has:

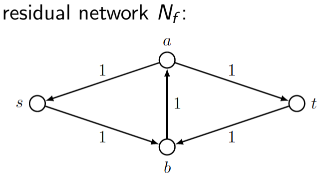
- For each $e \in E$ with $f(e) < c(e)$, an edge e in E_f with $c_f(e) \doteq c(e) - f(e)$.
- For each $e = (u, v) \in E$ with $f(e) > 0$, an edge $e' \doteq (v, u)$ in E_f with $c_f(e') \doteq f(e)$.

Unblocking example

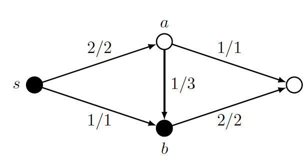
flow f:



residual network N_f:



Recap - properties



- Invariance: For every flow f and st-cut (S, T)
  $$f_{\text{out}}(S) - f_{\text{in}}(S) = \nu(f)$$
- Weak duality: For every flow f and st-cut (S, T)
  $$\nu(f) \leq c(S, T)$$
  Equality $\nu(f) = c(S, T)$ holds iff
  - $(\forall e \in E \cap S \times T)\, f(e) = c(e)$
  - $(\forall e \in E \cap T \times S)\, f(e) = 0$.
- Strong duality (today):
  $$\max_{\text{flow } f} \nu(f) = \min_{st\text{-cut } (S,T)} c(S, T)$$

Residual Network

Consider a flow f in $N = (V, E, c, s, t)$.

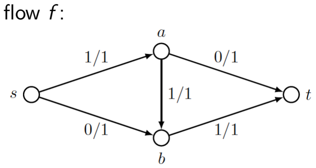
Definition

The residual network $N_f = (V, E_f, c_f, s, t)$ has:

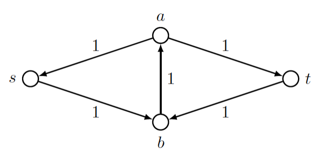
- For each $e \in E$ with $f(e) < c(e)$, an edge e in E_f with $c_f(e) \doteq c(e) - f(e)$.
- For each $e = (u, v) \in E$ with $f(e) > 0$, an edge $e' \doteq (v, u)$ in E_f with $c_f(e') \doteq f(e)$.

Unblocking example

flow f:



residual network N_f:



Path Augmentation

Schema

1. Start with $f \equiv 0$.
2. While there is an *st*-path in N_f
   - Pick such a path P.
   - $f \leftarrow f +$ flow along P of value $\min_{e \in P}(c_f(e))$
3. Return f.

Soundness

If the algorithm produces an output, it is correct.

- f always is a valid flow.
- If there is no *st*-path in N_f then $\nu(f)$ is maximized.

Termination

Soundness

Theorem

The following are equivalent:

(1) f has maximum value.
(2) There is no *st*-path in N_f.
(3) $\nu(f)$ equals the capacity of some *st*-cut.

Proof

(1) ⇒ (2) Contrapositive follows by path augmentation.
(2) ⇒ (3) Next slide.
(3) ⇒ (1) Follows from weak duality.

Corollary: Strong duality

$$\max_{\text{flow } f} \nu(f) = \min_{st\text{-cut } (S,T)} c(S, T)$$

Construction of Minimum Cut

Suppose that there is no *st*-path in N_f.

$$S \doteq \{v \in V : \text{ there exists an } sv\text{-path in } N_f\}.$$

- (S, T) with $T \doteq V \setminus S$ is an *st*-cut in N.
  - $s \in S$
  - $t \in T$
- $(\forall e \in E \cap S \times T)\, f(e) = c(e)$
  - Consider $e = (u, v) \in E$ with $u \in S$ and $f(e) < c(e)$.
  - Then $(u, v) \in E_f$ so $v \in S$.
- $(\forall e \in E \cap T \times S)\, f(e) = 0$
  - Consider $e = (u, v) \in E$ with $v \in S$ and $f(e) > 0$.
  - Then $e' \doteq (v, u) \in E_f$ so $u \in S$.
- $\therefore (S, T)$ is an *st*-cut with $c(S, T) = \nu(f)$.
- By weak duality $\nu(f)$ is maximized and $c(S, T)$ minimized.
- Construction of (S, T) from f runs in linear time.

Termination and Running Time

- Depend on the choice of augmenting path.
- There exist instances and choices without termination.

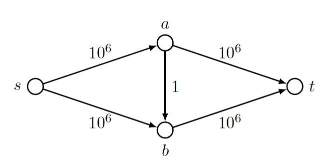
Integral capacities

If all capacities are integral, termination is guaranteed no matter how augmenting paths are picked.

- $\#$ augmentations $\leq F \doteq \max_{\text{flow } f}(\nu(f))$
- If each augmenting path is picked using linear-time graph traversal, running time is $O((n + m) \cdot F)$.
- Running time bounded by polynomial in:
  - size parameters (n and m)
  - *value* of numbers involved (capacities): $F \leq \sum_{e \in E} c(e)$

  Referred to as *pseudopolynomial* running time.

Slow convergence



Bad choices

Every augmentation through edge between a and b.

Good choices

- Augmenting path of maximum residual capacity
- Augmenting path with smallest number of edges

Better Algorithms

Augmentation along path of maximum residual capacity

- # augmentations $= O(m \cdot \log F)$ for integral instances
- Finding one path: $O(n+m)$ time
- Overall running time: $O((n+m)m \cdot \log F)$
- Running time is *polynomial*: bounded by polynomial in size parameters and *bitlength* of numbers involved.

Augmentation along path with smallest number of edges

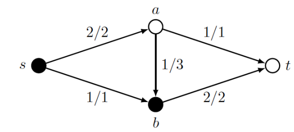
- # augmentations $= O(nm)$ for all instances
- Finding one path: $O(n+m)$ time (BFS)
- Overall running time: $O(nm(n+m))$
- Running time is *strongly polynomial*: bounded by polynomial in size parameters only.

Other approaches: $O(nm)$ time for all instances

Recap - notions

Integral network

- digraph (V, E)
- edge capacities $c : E \to \mathbb{N}$
- source $s \in V$ with indegree 0
- sink $t \in V$ with outdegree 0



Integral flow

A mapping $f : E \to \mathbb{N}$ satisfying

- [capacity constraints] $(\forall e \in E)\, f(e) \leq c(e)$
- [conservation constraints] $(\forall v \in V \setminus \{s,t\})\, f_{\text{in}}(v) = f_{\text{out}}(v)$
  where $f_{\text{in}}(v) \doteq \sum_{u \in V : e \doteq (u,v) \in E} f(e)$ and
  $f_{\text{out}}(v) \doteq \sum_{u \in V : e \doteq (v,u) \in E} f(e)$.

st-Cut

A partition (S, T) of V such that $s \in S$ and $t \in T$.

Bipartite Matching

Definition

A matching M in a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each $v \in V$ appears in at most one $e \in M$.

Computational problem

Input: bipartite graph $G = (V, E)$ with bipartition (L, R): $E \subseteq L \times R$ where $L \cap R = \emptyset$

Output: matching M such that $|M|$ is maximized

Reduction to max flow

- Model $e \in M$ as one unit of flow through e.
- Integral network N such that

$$\text{matching } M \text{ in } G \overset{\text{bijection}}{\longleftrightarrow} \text{integral flow } f \text{ in } N$$
$$|M| = \nu(f)$$

- Resulting algorithm runs in time $O(nm)$.

Bipartite Matching – duality

Capacity of a cut in N

- $c(S, T)$ is finite iff $E \cap S \times T = \emptyset$
  iff $G(L \cap S) \subseteq R \cap S$, (1)
  where $G(A) \doteq \{v : (\exists u \in A)\, (u, v) \in E\}$.
- If $c(S, T)$ is finite then $c(S, T) = |L \cap T| + |R \cap S|$. (2)

Matching obstacle

- Suppose not every $u \in L$ can be matched.
- $\nu(f) < |L|$ for every flow f in N.
- $c(S, T) < |L|$ for every min cut (S, T) in N. (3)
- Consider $A \doteq L \cap S$.

$$|G(A)| \overset{(1)}{\leq} |R \cap S| \overset{(2)}{=} c(S,T) - |L \cap T| \overset{(3)}{<} |L| - |L \cap T| = |L \cap S| = |A|$$

- Any $A \subseteq L$ with $|G(A)| < |A|$ is obstacle for matching all of L.

Marriage theorem

Definition

A matching M in $G = (V, E)$ is perfect if every $v \in V$ appears in some $e \in M$.

Marriage theorem [Hall]

A bipartite graph $G = (V, E)$ with bipartition (L, R) where $|L| = |R|$ has a perfect matching iff

$$(\forall A \subseteq L)\, |G(A)| \geq |A|.$$

Proof

$\Rightarrow$: Follows from definition of perfect matching.

$\Leftarrow$: Contrapositive follows from matching obstacle construction.

Edge-Disjoint Paths

Computational problem

Input: digraph $G = (V, E)$; $s, t \in V$

Output: set C of edge-disjoint *st*-paths in G such that $|C|$ is maximized

Reduction to max flow

- Model each path from s to t as one unit of flow.
- Integral network $N \doteq (V, E \setminus (V \times \{s\} \cup \{t\} \times V), c \equiv 1, s, t)$

$$\text{set } C \text{ of edge-disjoint } st\text{-paths in } G \longleftrightarrow \text{integral flow } f \text{ in } N$$
$$|C| = \nu(f)$$

- Resulting algorithm runs in time $O(nm)$.

Edge-Disjoint Paths – duality

Capacity of a cut in N

- $c(S,T) = |E \cap S \times T|$
- Removing $E \cap S \times T$ from G ensures no st-path remains.

Edge connectivity duality [Menger]

The maximum number of edge-disjoint st-paths equals the minimum number of edges to be removed so no st-path remains.

Proof

- $\ell \doteq$ max number of edge-disjoint st-paths
- $r \doteq$ min number of edges to be removed so no st-path
- $\max_{\text{flow } f}(\nu(f)) = \ell \leq r \leq \min_{st\text{-cut } (S,T)}(c(S,T))$
- By duality, LHS = RHS so $\ell = r$.

Survey Design

Computational problem

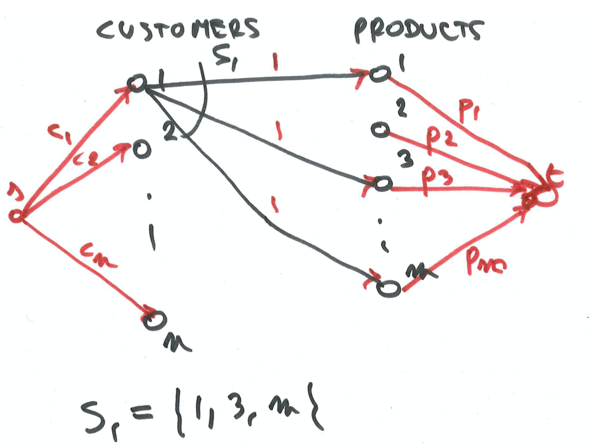
Input: n customers $i \in [n]$
m products $j \in [m]$
$S_i \subseteq [m]$: products that customer $i \in [n]$ can survey
$c_i \in \mathbb{N}$: max number of surveys for customer $i \in [n]$
$p_j \in \mathbb{N}$: min number of surveys of product $j \in [m]$

Output: set $D \subset [n] \times [m]$ such that
- $(\forall (i,j) \in D)\, j \in S_i$
- $(\forall i \in [n])\, |\{j \in [m] : (i,j) \in D\}| \leq c_i$
- $(\forall j \in [m])\, |\{i \in [m] : (i,j) \in D\}| \geq p_j$

Model

Represent each $(i,j) \in D$ as a unit of flow that passes through customer i and product j.

Survey Design – reduction to max flow



Survey Design – analysis

- Minimal survey D satisfies
$$(\forall j \in [m])\, |\{i \in [n] : (i,j) \in D\}| = p_j.$$
- Integral network N such that
$$\text{minimal survey } D \overset{\text{bijection}}{\longleftrightarrow} \text{integral flow } f \text{ with } \nu(f) = \sum_{j \in [m]} p_j$$
$$D = \text{middle edges that carry flow}$$
- Resulting algorithm runs in time $O((n+m)(n+m+\sum_{i\in[n]}|S_i|))$.

Image segmentation

Computational problem

Input: grid of pixels $i \in [n]$
$f_i \in [0,\infty)$: "likelihood" that i is foreground
$b_i \in [0,\infty)$: "likelihood" that i is background
$c \in [0,\infty)$: penalty for separating neighboring pixels

Output: partition of $[n]$ into foreground F and background B maximizing
$$\sum_{i\in F} f_i + \sum_{j \in B} b_j - c \cdot |\{(i,j) \in F \times B : i \sim j\}|$$

Model

- Vertex for each pixel, source s, sink t
- $S = F \cup \{s\}$ and $T = B \cup \{t\}$

Image Segmentation – rewriting objective

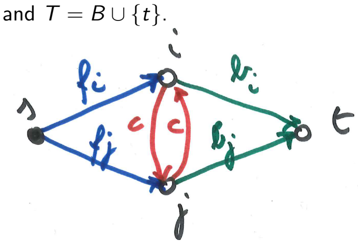
$$\max_{F,B}\left(\sum_{i\in F} f_i + \sum_{j\in B} b_j - \sum_{\substack{i\sim j\\(i,j)\in F\times B}} c\right)$$
$$= -\min_{F,B}\left(\sum_{\substack{i\sim j\\(i,j)\in F\times B}} c - \sum_{i\in F} f_i - \sum_{j\in B} b_j\right)$$
$$= -\min_{F,B}\left(\sum_{\substack{i\sim j\\(i,j)\in F\times B}} c - \sum_{i\in[n]} f_i + \sum_{i\in B} f_i - \sum_{j\in[n]} b_j + \sum_{j\in F} b_j\right)$$
$$= \sum_{i\in[n]} f_i + \sum_{j\in[n]} b_j - \min_{F,B}\left(\sum_{\substack{i\sim j\\(i,j)\in F\times B}} c + \sum_{i\in B} f_i + \sum_{j\in F} b_j\right)$$

Image Segmentation – reduction to min cut

- Need to find partition of $[n]$ into F and B minimizing

$$(*) \doteq \sum_{\substack{i \sim j \\ (i,j) \in F \times B}} c + \sum_{i \in B} f_i + \sum_{j \in F} b_j.$$

- Construct network N such that $(*) = c(S, T)$ where $S = F \cup \{s\}$ and $T = B \cup \{t\}$.



- Resulting algorithm runs in time $O(n^2)$.

Project Selection

Computational problem

Input: n projects $i \in [n]$
m tools $j \in [m]$
$T_i \subseteq [m]$: tools needed to realize project $i \in [n]$
$v_i \in [0, \infty)$: value of project $i \in [n]$ if realized
$c_j \in [0, \infty)$: one-time cost of tool $j \in [m]$ if bought

Output: Set of projects $I \subseteq [n]$ to realize and set of tools $J \subseteq [m]$ to buy maximizing $\sum_{i \in I} v_i - \sum_{j \in J} c_j$ such that $(\forall i \in I)\ T_i \subseteq J$.

Model

- Vertex for each project $i \in [n]$ & tool $j \in [m]$; source s, sink t
- Project $i \in [n]$ is realized iff $i \in S$.
- Side of st-cut determines whether tool $j \in [m]$ is bought.

Project Selection – rewriting objective

$$\max_{I,J} \left(\sum_{i \in I} v_i - \sum_{j \in J} c_j \right)$$

$$= -\min_{I,J} \left(\sum_{j \in J} c_j - \sum_{i \in I} v_i \right)$$

$$= -\min_{I,J} \left(\sum_{j \in J} c_j - \sum_{i \in [n]} v_i + \sum_{i \in [n] \setminus I} v_i \right)$$

$$= \sum_{i \in [n]} v_i - \min_{I,J} \left(\sum_{j \in J} c_j + \sum_{i \in [n] \setminus I} v_i \right)$$

Project Selection – reduction to min cut

- Need to find $I \subseteq [n]$ and $J \subseteq [m]$ with $(\forall i \in I)\ T_i \subseteq J$ minimizing

$$(*) \doteq \sum_{j \in J} c_j + \sum_{i \in [n] \setminus I} v_i.$$

- Construct network N such that $(*) = c(S, T)$ where I are the projects in S.
- Let J be the tools in S.
- Enforce condition $(\forall i \in I)\ T_i \subseteq J$ by including edges (i, j) with $c(i, j) = \infty$ for each $i \in [n]$ and $j \in T_i$
- Resulting algorithm runs in time $O((n + m)(n + m + \sum_{i \in [n]} |T_i|))$.

Project Selection – reduction to min cut