Mathematics 522
Spring 2023
Homework assignment No. 7.

Due March 28.

• Exercises on page 77 (on differential equations): 3.72, 3.73, 3.74.

In addition do the following problem.

1. Let $I = [a, b]$, assume $0 < s \le 1$ and let $\mathrm{Lip}_s(I)$ be the class of continuous functions f with the following property: *There is a number C such that for all $x \in I$, $x' \in I$ the inequality*

$$|f(x) - f(x')| \le C|x - x'|^s;$$

holds; here C depends on the function.

(i) Show that $\mathrm{Lip}(s)$ is a vector space and

$$\|f\|_{\mathrm{Lip}_s} = \sup_{x \in I} |f(x)| + \sup_{\substack{x, x' \in I \\ x \ne x'}} \frac{|f(x) - f(x')|}{|x - x'|^s}$$

defines a norm on this space.

(ii) Show that Lip_s with the norm in (i) is a complete vector space.

(iii) Let $I = [0, 1]$. Consider for $0 < \gamma < 1$ the function $u_\gamma(x) = x^\gamma$ on I. Show that $u_\gamma \in \mathrm{Lip}_s(I)$ if and only if $s \le \gamma$.

(iv) One might ask why we restrict the choice of s to $s \le 1$. To answer this let us define $\mathrm{Lip}(s)$ for $s > 1$ as in part (i). Prove: the resulting vector space only contains the functions that are constant on I.

(v) Show that $C^1(I) \subset \mathrm{Lip}_1(I)$ and give an example of a function in $\mathrm{Lip}_1(I)$ which is not differentiable at some point.

Terminology: The space Lip_s is sometimes called the space of *s-Lipschitz continuous functions on I*, and sometimes called the space of *s-Hölder continuous functions* on I.

2. Consider the space $C([0, 1])$ of continuous functions on $[0, 1]$.

(i) Show that $\|f\|_1 = \int_0^1 |f(x)| dx$ defines a norm on $C[0, 1]$.

(ii) Define

$$f_n(x) = \begin{cases} x^{-1/2} & \text{for } n^{-1} \le x \le 1 \\ n^{1/2} & \text{for } 0 \le x \le n^{-1} \end{cases}$$

In tthis normed space show that f_n is a Cauchy-sequence which does not converge to an $f \in C([0, 1])$.