pairwise disjoint events $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ (event decomposition)

for any $A \subset \Omega$, $P(A) = \frac{\#A}{\#\Omega}$ (fair)

sampling

- with replacement, order matters $\#\Omega = n^k$
- without replacement, order matters $\#\Omega = n \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!} = (n)_k$
- without replacement, order irrelavant $\#\Omega = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$

negative binomial $X \sim Negbin(\underbrace{k}_{\text{no. successes}}, \underbrace{p}_{\text{success param}})$ $X \in \{k, k+1, \dots\}$

$P(X=n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$ $(\forall n \geq k)$ ← n: no. trails

$0 < p \leq 1$, $k \geq 1$

Consequences of rule of probability

- event decomposition
- complement $P(A) + P(A^c) = 1$ $(A, A^c$ disjoint$)$
- monotonicity $A \subseteq B \Rightarrow P(A) \leq P(B)$, $B = A \cup A^c B$

Inclusion-exclusion $P(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} P(A_{i_1} \cap \dots \cap A_{i_k})$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Conditional probability $P(A|B) = \frac{P(AB)}{P(B)}$ $(B \subset \Omega, P(B) > 0)$

multiplication $P(A_1 A_2 \dots A_n) = P(A_1) P(A_2|A_1) \cdots P(A_n | A_1 \cdots A_{n-1})$

partition $B_i B_j = \emptyset$ $(i \neq j)$ $\bigcup_{i=1}^{n} B_i = \Omega$ for $\{B_1, \dots, B_n\}$

$P(A) = \sum_{i=1}^{n} P(AB_i) = \sum_{i=1}^{n} P(A|B_i) P(B_i)$

Bayes' formula $P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$ $P(A), P(B), P(B^c) > 0$

$P(B_k|A) = \frac{P(AB_k)}{P(A)} = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$ $P(A), P(B_i) > 0$, B partition Ω

Independent A, B independent if $P(AB) = P(A)P(B)$

mutually independent $P(A_{i_1} A_{i_2} \dots A_{i_n}) = P(A_{i_1}) P(A_{i_2}) \dots P(A_{i_n})$

conditionally independent given B $P(A_{i_1} A_{i_2} \dots A_{i_k} | B) = P(A_{i_1}|B) P(A_{i_2}|B) \cdots P(A_{i_k}|B)$ $P(B) > 0$

Distribution

Bernoulli $X \sim Ber(\underbrace{p}_{\text{success parameter}})$ $0 \leq p \leq 1$, $X \in \{0, 1\}$, $P(X=1) = p$, $P(X=0) = 1-p$

Negative Binomial ↑

Binomial $X \sim Bin(\underbrace{n}_{\text{number of trails}}, \underbrace{p}_{\text{success parameter}})$ $0 \leq p \leq 1$, $X \in \{0, 1, \dots, n\}$, $\underbrace{k}_{\text{number of successes}} = 0, 1, \dots, n$

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

(continuous)

Uniform $X \sim Unif[a,b]$

$[a,b]$ real line

p.d.f.

$f(x) = \begin{cases} \frac{1}{b-a} & (x \in [a,b]) \\ 0 & (x \notin [a,b]) \end{cases}$

Geometric $X \sim Geom(\underbrace{p}_{\text{success parameter}})$ $0 < p \leq 1$, $X \in \{1, 2, 3, \dots\}$, $\underbrace{k}_{\text{number of trails needed to see first success}} \in \mathbb{Z}^+$

$P(X=k) = (1-p)^{k-1} p$

$p = 0 \Rightarrow P(N=k) = 0$, $P(N=\infty) = 1$

$\sum_{k=1}^{\infty} P(N=k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = 1$

Hypergeometric $X \sim Hypergeom(\underbrace{N}_{\text{no. all elements}}, \underbrace{N_A}_{\text{no. elements desired category}}, \underbrace{n}_{\text{no. trails}})$ $0 \leq N_A \leq N$, $1 \leq n \leq N$, $X \in \{0, 1, \dots, n\}$, $\underbrace{k}_{\text{no. occurence desired category in trails}} = 0, 1, \dots, n$

$P(X=k) = \frac{\binom{N_A}{k}\binom{N-N_A}{n-k}}{\binom{N}{n}}$

p.m.f. for discrete r.v. X $\quad P(X\in A)=1 \quad P(X=k)>0 \quad p_X(k)=P(X=k)$

$\sum_k p_X(k)=1 \quad P(X\in B)=\sum_{k\in B} p_X(k)$ for $B\subset\mathbb{R}$

p.d.f. for r.v. X if f satisfies $P(X\le b)=\int_{-\infty}^{b} f(x)\,dx \quad \forall b\in\mathbb{R}$

(f: density function) facts: $P(X\in B)=\int_B f(x)\,dx$ B real line, $P(X=c)=0\ (\forall c\in\mathbb{R})$

Infinitesimal continuous at a, small $\varepsilon>0$ $\quad P(a<X<a+\varepsilon)\approx f(a)\cdot\varepsilon$

c.d.f for r.v. X $\quad F(s)=P(X\le s)\ \forall s\in\mathbb{R}$ $\quad$ fact: $P(a<X\le b)=P(X\le b)-P(X\le a)=F(b)-F(a)$

discrete $=\sum_{k:k\le s} P(X=k)$ $\quad$ fact: F pairwise constant; $P(X=x)$ is magnitude of jump of F at x

continuous $=\int_{-\infty}^{s} f(x)\,dx$ $\quad$ fact: $f(x)=F'(x)$; if F not differentiable at x, $f(x)$ is arbitrary

properties monotonicity: $s<t \Rightarrow F(s)\le F(t)$ $\quad$ right continuity: $F(t)=\lim_{s\to t^+} F(s)\ \forall t\in\mathbb{R}$

$\lim_{t\to-\infty} F(t)=0$, $\lim_{t\to\infty} F(t)=1$, $P(X<a)=\lim_{s\to a^-} F(s)\ \forall a\in\mathbb{R}$

Expectation/mean of discrete r.v. X $\quad E(X)=\sum_k kP(X=k) \quad E[g(X)]=\sum_k g(k)p_X(k)$

$E(aX+b)=aE[X]+b$

continuous $\quad E(X)=\int_{-\infty}^{\infty} x f(x)\,dx,\ E[g(X)]=\int_{-\infty}^{\infty} g(x)f(x)\,dx$

Variance $\quad Var(X)=E[(X-E(X))^2]=E[X^2]-(E[X])^2 \quad Var(aX+b)=a^2 Var(X)$

Series $\quad \frac{1}{1-x}=\sum_{n=0}^{\infty} x^n\ (|x|<1) \quad \frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n\ (|x|<1) \quad \ln(1+x)=\sum_{\substack{x=1\\2n}}^{\infty}(-1)^{n-1}\frac{x^n}{n}\ (-1<x\le 1)$

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ (\forall x\in\mathbb{R}) \quad \sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \quad \cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x}{(2n)!}$

$(1+x)^p=\sum_{k=0}^{\infty}\binom{p}{k}x^k=\sum_{k=0}^{\infty}\frac{p(p-1)\cdots(p-k+1)}{k!}x^k \quad (|x|<1,\ p\in\mathbb{R})$

$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k=\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$

$$\boxed{\begin{array}{ll} \sum_{n=1}^{\infty} ar^{n-1}=\frac{a}{1-r} & (|r|<1) \\ \sum_{n=1}^{k} ar^{n-1}=a\left(\frac{1-r^{k+1}}{1-r}\right) & (|r|<1) \\ r=\left(\frac{a_{n+1}}{a_n}\right) & \text{geometric series} \end{array}}$$

Integral by substitution $\int f(g(x))g'(x)\,dx=\int f(u)\,du$ $\quad$ "$+C$"

by parts $\int u\,dv=uv-\int v\,du$

Taylor at $x=c$: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$

$\int u^n\,du=\frac{u^{n+1}}{n+1}\ (n\ne -1) \quad \int u^{-1}\,du=\ln|u|$

$\int e^u\,du=e^u \quad \int a^u\,du=\frac{a^u}{\ln u} \quad \int \ln u\,du=u\ln u-u \quad \int \log_a u\,du=\frac{u\ln u-u}{\ln a}$

$\int \cos u\,du=\sin u \quad \int \sin u\,du=-\cos u \quad \int \sec^2 u\,du=\tan u \quad \int \csc^2 u\,du=-\cot u$

Derivative $\quad \lim_{x\to 0}\frac{\sin x}{x}=1 \quad f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$

$(\sin x)'=\cos x \quad (\cos x)'=-\sin x \quad (\tan x)'=\sec^2 x \quad (\cot x)'=-\csc^2 x$

$(e^x)'=e^x \quad (a^x)'=a^x\ln a \quad (\ln x)'=\frac{1}{x} \quad (\log_a x)'=\frac{1}{x\ln a} \quad (x^x)'=x^x(\ln x+1)$

MATH431 Midterm 2 Note

Ruixuan Tu

Random Variables:

Discrete	p.m.f. $p_X(k)=P(X=k)$	c.d.f. $F_X(a)=P(X\le a) = \sum_{k: k\le a} p_X(k)$; $E(X)=\sum_k k\,p_X(k)$
Continuous	p.d.f. $f_X(x)$	c.d.f. $F_X(a)=P(X\le a) = \int_{-\infty}^{a} f(x)\,dx$; $E(X)=\int_{-\infty}^{\infty} x f(x)\,dx$

Discrete	$E[g(X)] = \sum_k g(k)\,p_X(k)$	$\sigma^2 = Var(X) = E[(X-\mu)^2] = \sum_k (k-\mu)^2 p_X(k)$
Continuous	$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$	$\sigma^2 = Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$SD(X)=\sigma=\sqrt{Var(X)}$ $\quad Var(X)=E[(X-E[X])^2]=E[X^2]-(E[X])^2$ $\quad E(aX+b)=aE[X]+b$

$Var(aX+b)=a^2\,Var(X)$

Normal Distribution: Standard $Z\sim Normal(0,1)$ $\quad \varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ $\quad E(Z)=0,\ Var(Z)=1$

$X\sim Normal(\mu,\sigma^2)$ $\quad f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\quad Z=\frac{X-\mu}{\sigma}$

$X\sim Normal(\mu,\sigma^2),\ Y=aX+b \Rightarrow Y\sim Normal(a\mu+b, a^2\sigma^2)$ $\quad \Phi(-z)=1-\Phi(z)$

Approximations for Binomial: CLT $S_n\sim Bin(n,p)$, $np(1-p)>10 \Rightarrow P\left(a\le \frac{S_n-np}{\sqrt{np(1-p)}}\le b\right) \underset{\text{close to}}{\overset{(n\to\infty \text{ for CLT})}{=}} \Phi(b)-\Phi(a)$ by $X\sim Normal(np, np(1-p))$

Continuity correction $P(k_1\le n\le k_2)=P(k_1-1/2\le S_n\le k_2+1/2)$

LLN $\varepsilon>0$, $\lim_{n\to\infty} P\left(\left|\frac{S_n}{n}-p\right|<\varepsilon\right)=1$ $\quad \frac{S_n}{n}\approx p+\frac{\sigma}{\sqrt{n}}Z$ where $\sigma=\sqrt{p(1-p)}$

Confidence Interval $P(|\hat{p}-p|<\varepsilon)\ge 2\Phi(2\varepsilon\sqrt{n})-1$

Random Walk $P(S_{2n}=0)=\binom{2n}{n}p^n(1-p)^{2n-n}$

Poisson Distribution: $\lambda>0$, $P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}$ for $X\sim Poisson(\lambda)$, $k\in\{0,1,2,\dots\}$

for $S_n\sim Bin(n,\lambda/n)$ $\lim_{n\to\infty}P(S_n=k)=P(X=k)$ $\quad$ condition: $np^2<<1$ or $np^2\le\frac{1}{10}$

Let $X\sim Bin(n,p)$, $Y\sim Poisson(np)$, for subset $A\subseteq\{0,1,2,\dots\}$, $|P(X\in A)-P(Y\in A)|\le np^2$

Exponential Distribution: $0<\lambda<\infty$, $f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\ge 0\\ 0, & x<0\end{cases}$ for $X\sim Exp(\lambda)$ (rate) $\quad F(x)=1-e^{-\lambda x}, x\ge 0$ and $0, x<0$

Memoryless property $P(X>t+s\mid X>t)=P(X>s)$

for $nT_n\sim Geom(\lambda/n)$, $\lambda>0$, n large that $\lambda/n<1$, $\lim_{n\to\infty}P(T_n>t)=e^{-\lambda t}$, $t\ge 0$

Transforms and Transformations: MGF $M(t)=E(e^{tX})$ $\quad$ Moment $E(X^n)=M^{(n)}(0)$ $\quad X\overset{d}{=}Y$ iff $P(X\in B)=P(Y\in B)$ for all $B\subseteq\mathbb{R}$ (def)

$F_Y(y)=P(Y\le y)=P(g(X)\le y)$ for $Y=g(X)$ $\quad$ CHAIN RULE $\quad$ if $\exists\delta>0$, $t\in(-\delta,\delta)$, $M_X(t)=M_Y(t)$ (fact)

Joint distribution of random variables:

Discrete $\quad p(k_1,k_2,\dots,k_n)=P(X_1=k_1, X_2=k_2,\dots,X_n=k_n)$ joint pmf

sum well defined, $g:\mathbb{R}^n\to\mathbb{R}$, $E[g(X_1,\dots,X_n)]=\sum_{k_1,\dots,k_n} g(k_1,\dots,k_n)\,p(k_1,\dots,k_n)$

$p_{X_j}(k)=\sum_{l_1,\dots,l_{j-1},l_{j+1},l_n} p(l_1,\dots,l_{j-1},k,l_{j+1},\dots,l_n)$ marginal pmf of X_j

$p_{X_1,\dots,X_m}(k_1,\dots,k_m)=\sum_{l_{m+1},\dots,l_n} p(k_1,\dots,k_m,l_{m+1},\dots,l_n)$

Multinomial Distribution: $(X_1,\dots,X_r)\sim Multi(n,r,p_1,\dots,p_r)$ $\quad p_1+\dots+p_r=1$

$P(X_1=k_1,X_2=k_2,\dots,X_r=k_r)=\binom{n}{k_1,k_2,\dots,k_r}p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ $\quad k_i\ge 0,\ k_1+\dots+k_r=n$

Continuous $\quad P((X_1,\dots,X_n)\in B)=\int\cdots\int_B f(x_1,\dots,x_n)\,dx_1\cdots dx_n$ jointly continuous / joint df

$\dots$, $E[g(X_1,\dots,X_n)]=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\dots,x_n)f(x_1,\dots,x_n)\,dx_1\cdots dx_n$

$f_{X_j}(x)=\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n-1\text{ integrals}} f(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_n)\,dx_1\cdots dx_{j-1}\,dx_{j+1}\cdots dx_n$

for X,Y: $f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

Uniform Distribution: $D\subseteq\mathbb{R}^2$ $\quad f(x,y)=\begin{cases}\frac{1}{area(D)}, & (x,y)\in D\\ 0, & (x,y)\notin D\end{cases}$

$B\subseteq\mathbb{R}^3$ $\quad f(x,y,z)=\begin{cases}\frac{1}{vol(B)}, & (x,y,z)\in B\\ 0, & (x,y,z)\notin B\end{cases}$

Independence discrete independent iff $p(k_1,\dots,k_n)=p_{X_1}(k_1)\cdots p_{X_n}(k_n)$

jointly continuous iff independent iff $f(x_1,x_2,\dots,x_n)=f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n)$

independent iff $F(x_1,x_2,\dots,x_n)=\prod_{k=1}^{n}F_{X_k}(x_k)$

Distributions — Discrete

Bernouli: $0 \le p \le 1$; $p_X(0) = 1-p$, $p_X(1) = p$; $E(X) = p$; $Var(X) = p(1-p)$

Binomial: $0 \le p \le 1$, $n \ge 1$; $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \le k \le n$; $E(X) = np$; $Var(X) = np(1-p)$; k: no. successes

Geometric: $0 < p \le 1$; $p_X(k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$; $E(X) = \frac{1}{p}$; $Var(X) = \frac{1-p}{p^2}$; k: no. trails needed to see first success

Poisson: $\lambda > 0$; $p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, 2, \ldots$; $E(X) = \lambda$; $Var(X) = \lambda$

Hypergeometric: $N \ge 1$, $N_A \ge 0$, $n \ge 1$; $p_X(k) = \frac{\binom{N_A}{k}\binom{N-N_A}{n-k}}{\binom{N}{n}}$, $0 \le k \le n$; $E(X) = \frac{nN_A}{N}$; $Var(X) = \frac{N-n}{N-1} n \frac{N_A(N-N_A)}{N^2}$

N: no. all elements　n: trails　k: times desired category in trails　N_A: no. all elements of desired category

Negative binom: $k \ge 1$, $0 < p \le 1$; $p_X(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$, $n \le k$; $E(X) = \frac{k}{p}$; $Var(X) = k \cdot \frac{1-p}{p^2}$

Continuous

Uniform: $a < b$; $f_X(t) = \frac{1}{b-a}$ for $t \in [a,b]$; $E(X) = \frac{a+b}{2}$; $Var(X) = \frac{(b-a)^2}{12}$

Normal: $\mu \in \mathbb{R}, \sigma^2 > 0$; $f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$; $E(X) = \mu$; $Var(X) = \sigma^2$

Exponential: $\lambda > 0$; $f_X(t) = \lambda e^{-\lambda t}$ for $t \ge 0$; $E(X) = \frac{1}{\lambda}$; $Var(X) = \frac{1}{\lambda^2}$

$F_X(t) = 1 - e^{-\lambda x}$

Probability

Sampling
- w/ replacement, order matters n^k
- wo/ replacement, order matters $n \cdot \cdots \cdot (n-k+1) = \frac{n!}{(n-k)!} = (n)_k$
- wo/ replacement, order irrelavant $\frac{n!}{(n-k)!\,k!} = \binom{n}{k}$

Inclusion-exclusion

$P(A \cup B) = P(A) + P(B) - P(AB)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

Condition　Bayes: $P(B|A) = \frac{P(AB)}{P(A)}$　independent: $P(AB) = P(A)P(B)$

Calculus

Multivar

Polar Coordinates $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$ – $\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Change Var $x = g(u,v)$, $y = h(u,v)$, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v))\, |J|\, du\, dv$

Series

Geometric $\forall |r| < 1$, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, $\sum_{n=1}^{k} ar^{n-1} = a\left(\frac{1-r^{k+1}}{1-r}\right)$, $r = \left(\frac{a_{n+1}}{a_n}\right)$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $(|x| < 1)$; $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $(x \in \mathbb{R})$; $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ $(|x| < 1)$　Taylor at $x = c$: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

Integral

by substitution $\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$

by parts $\int u\, dv = uv - \int v\, du$

$\int u^n du = \frac{u^{n+1}}{n+1}$ $(n \ne -1)$; $\int u^{-1} du = \ln|u|$; $\int e^u du = e^u$; $\int a^u du = \frac{a^u}{\ln a}$; $\int \ln u\, du = u\ln u - u$

$\int \log_a u\, du = \frac{u\ln u - u}{\ln a}$

Derivative

$\lim_{x \to 0} \frac{\sin x}{x} = 1$; $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$; $(\sin x)' = \cos x$; $(\cos x)' = -\sin x$

$(e^x)' = e^x$; $(a^x)' = a^x \ln a$; $(\ln(x))' = \frac{1}{x}$; $(\log_a x)' = \frac{1}{x\ln a}$; $f(x) \cdot g(x) = f'(x)g(x) + f(x)g'(x)$

Basic　一元二次方程 $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$　二次函数 顶点式 $y = a(x-h)^2 + k$ at (h,k) 对称轴 $x = h$, 开○ ax^2

交点式 $y = a(x - x_1)(x - x_2)$ at $(x_1, 0)$, $(x_2, 0)$

Gamma: $r \ge 1$, $\lambda > 0$; $f_X(t) = \frac{\lambda^r x^{r-1}}{\Gamma(r)} e^{-\lambda t}$ for $t \ge 0$; $E(X) = \frac{r}{\lambda}$, $Var(X) = \frac{r}{\lambda^2}$; $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$, $\Gamma(n) = (n-1)!$

MATH 431 Final Note

Sums and Symmetry

$p_{X+Y}(n) = p_X * p_Y(n) = \sum_k p_X(k)\,p_Y(n-k) = \sum_\ell p_X(n-\ell)\,p_Y(\ell)$ with X, Y independent r.v.s (fact)

$f_{X+Y}(z) = f_X * f_Y(z) = \int_{-\infty}^{\infty} f_X(x)\,f_Y(z-x)\,dx = \int_{-\infty}^{\infty} f_X(z-x)\,f_Y(x)\,dx$

$X \sim Negbin(k,p)$ $\quad P(X=n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$ for $n \ge k$, $X \in \{k, k+1, k+2, \dots\}$ (Def)

X: no. trials to get k successes $\quad Negbin(1,p) \stackrel{d}{=} Geom(p)$

$\mu = a_1\mu_1 + \dots + a_n\mu_n + b$, $\sigma^2 = a_1^2\sigma_1^2 + \dots + a_n^2\sigma_n^2$ with $X_i \sim Normal(\mu_i, \sigma_i^2)$, $a_i \ne 0$, $b \in \mathbb{R}$, $X \sim Normal(\mu, \sigma^2)$

Exchangeable $(X_1, X_2, \dots, X_n) \stackrel{d}{=} (X_{k_1}, X_{k_2}, \dots, X_{k_n})$ with $(X_{k_1}, X_{k_2}, \dots, X_{k_n})$ be permutation of $(X_1, X_2, \dots, X_n)$ (def)

iff $p_{X_1,\dots,X_n}$ or $f(x_1, \dots, x_n)$ a symmetric function (fact)

then X_k are identically distributed (same marginal distribution), and $E[g(X_1,\dots,X_k)] = E[g(X_{i_1},\dots,X_{i_k})]$ $(1 \le k \le n)$ (fact)

IID independent and identically distributed (X_k has same p.d.f.) (def) $\Rightarrow$ exchangeable (Thm)

$1 \le k \le n$, $X_1, \dots, X_k$ outcomes of success draws w/ replacement from $\{x_1, \dots, x_n\} \Rightarrow X_1, \dots, X_k$ i.i.d. (Thm)

$X_1, \dots, X_n$ exchangeable $\Rightarrow g(X_1), \dots, g(X_n)$ exchangeable for any function g (Thm)

(Waiting time for Poisson process) Let $0 < T_1 < T_2 < \dots < T_n$ be positions of random points on $[0, \infty]$, $T_n - T_{n-1}, \dots, T_2 - T_1, T_1$ i.i.d. with $T_i - T_{i-1} \sim Exp(\lambda)$

sum of n independent $Exp(\lambda)$ vars, i.e. $n \cdot X_i \sim Gamma(n, \lambda)$ with $X_i \sim Exp(\lambda)$

Expectation and Variance in Multivariate

$E[g_1(X_1) + \dots + g_n(X_n)] = E[g_1(X_1)] + \dots + E[g_n(X_n)]$, $E[X_1 + \dots + X_n] = E[X_1] + \dots + E[X_n]$ (fact, linearity of E)

$E\left[\prod_{k=1}^n g_k(X_k)\right] = \prod_{k=1}^n E[g_k(X_k)]$, $Var(X_1 + \dots + X_n) = Var(X_1) + \dots + Var(X_n)$ $\quad X_i$ independent r.v.s

$E[\bar{X}_n] = \mu$, $Var(\bar{X}_n) = \frac{\sigma^2}{n}$ with $\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$ and X_i i.i.d. r.v.s

$M_{X+Y}(t) = M_X(t)M_Y(t)$, X and Y ind r.v. with $M_X(t)$ and $M_Y(t)$

$Cov(X,Y) \stackrel{def}{=} E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y$ $\quad$ correlated $\begin{cases} \text{positively if } Cov(X,Y) > 0 \\ \text{negatively if } Cov(X,Y) < 0 \\ \text{un- if } Cov(X,Y) = 0 \end{cases}$ $\quad$ independent $\Rightarrow$ uncorrelated

$Var\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n Var(X_i) + 2\sum_{1 \le i < j \le n} Cov(X_i, X_j)$

$Cov(X,Y) = Cov(Y,X)$, $Cov(aX+b, Y) = a\,Cov(X,Y)$, $Cov\left(\sum_{i=1}^m a_iX_i, \sum_{j=1}^n b_jY_j\right) = \sum_{i=1}^m \sum_{j=1}^n a_ib_j\,Cov(X_i, Y_j)$

$\rho(X,Y) = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$ $\quad Corr(aX+b, Y) = \frac{a}{|a|}Corr(X,Y)$, $-1 \le Corr(X,Y) \le 1$, $Corr(X,Y) = 1$ iff $\exists a > 0, b \in \mathbb{R}, Y = aX + b$

$Corr(X,Y) = -1$ iff $\exists a < 0, b \in \mathbb{R}, Y = aX + b$

Tail Bounds and Limit Thms

$X \ge Y \Rightarrow E[X] \ge E[Y]$

Continuity for mgf: $\lim_{n\to\infty} M_{Y_n}(t) = M_Y(t) \Rightarrow \lim_{n\to\infty} P(Y_n \le a) = P(Y \le a)$ in some $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$; $= e^{t^2/2} \Rightarrow \lim_{n\to\infty} P(Y_n \le a) = \Phi(a)$

Markov's inequality $P(X \ge c) \le \frac{E[X]}{c}$ with X nonnegative r.v., $\forall c > 0$

Chebyshev's inequality $P(|X-\mu| \ge c) \le \frac{\sigma^2}{c^2}$ with X r.v. with μ, Var σ^2, $\forall c > 0$

LLN $\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) = 1$ with $\varepsilon > 0$, X_i i.i.d. r.v.'s $E[X_1] = \mu$, $Var[X_1] = \sigma^2$, $S_n = X_1 + \dots + X_n$

CLT $\lim_{n\to\infty} P\left(a \le \frac{S_n - n\mu}{\sigma\sqrt{n}} \le b\right) = \Phi(b) - \Phi(a)$ with X_i i.i.d. rvs; $\lim_{n\to\infty} P\left(a \le \frac{(\bar{X}_n - \mu)\sqrt{n}}{S_n} \le b\right) = \Phi(b) - \Phi(a)$

CI $P(\bar{X}_n - \varepsilon \le \mu \le \bar{X}_n + \varepsilon) = P(\mu - \varepsilon \le \bar{X}_n \le \mu + \varepsilon) = P\left(-\frac{\varepsilon\sqrt{n}}{S_n} \le \frac{(\bar{X}_n - \mu)\sqrt{n}}{S_n} \le \frac{\varepsilon\sqrt{n}}{S_n}\right) \approx 2\Phi\left(\frac{\varepsilon\sqrt{n}}{S_n}\right) - 1$ ← confidence

Strong LLN $P\left(\lim_{n\to\infty} \frac{X_1 + \dots + X_n}{n} = \mu\right) = 1$ $\quad$ iid X_i, $E[X_i] = \mu$

Conditional Distribution

Discrete $p_{X|B}(k) = P(X=k|B) = \frac{P(\{X=k\} \cap B)}{P(B)}$, $E[X|B] = \sum_k k\,p_{X|B}(k) = \sum_k k\,P(X=k|B)$ (def)

$p_X(k) = \sum_{i=1}^n p_{X|B_i}(k)P(B_i)$, $E[X] = \sum_{i=1}^n E[X|B_i]P(B_i)$ with $B_1 \dots B_n$ partition of Ω

$p_{X|Y}(x|y) = P(X=x|Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$, $E[X|Y=y] = \sum_x x\,p_{X|Y}(x|y)$ (def)

$p_X(x) = \sum_y p_{X|Y}(x|y)p_Y(y)$, $E(X) = \sum_y E[X|Y=y]p_Y(y)$

Continuous $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$, $P(X \in A|Y=y) = \int_A f_{X|Y}(x|y)\,dx$, $E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\,dx$ (def)

$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)\,dy$, $E[g(X)] = \int_{-\infty}^{\infty} E[g(X)|Y=y] f_Y(y)\,dy$ $\quad X, Y$ independent iff $p_{X|Y}(x|y) = p_X(x)$ or $f_{X|Y}(x|y) = f_X(x)$

Expectation $E(X|Y)$ is r.v. $v(Y)$ with $v(y) = E[X|Y=y]$ (def) $\quad E[g_1(X_1) + \dots + g_n(X_n)|Y] = E[g_1(X_1)|Y] + \dots + E[g_n(X_n)|Y]$, $E(aX+b|Y) = aE(X|Y) + b$

$E[E[g(X_1, \dots, X_n)|Y]] = E[g(X_1, \dots, X_n)]$

Further $E[X_1 + \dots + X_N] = E[S_N] = E[N] \cdot E[X_1]$ $\quad X_i$ i.i.d.

Markov chain $P(X_{n+1} = x_{n+1} | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) = P(X_{n+1} = x_{n+1} | X_n = x_n)$