MATH 521 Lecture 10

INNER PRODUCT SPACE

Def An <u>inner product space</u> is a vector space V with a function $\langle \ , \ \rangle : V \times V \to \mathbb{R}$ such that

- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$
- $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle \quad x, y \in V \quad \lambda \in \mathbb{R}$
- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$

  (positive definite)

E.G.

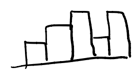
- $V = \mathbb{R} \quad \langle x, y\rangle = xy$ $\qquad \langle v, v\rangle = \sum x_i^2 = \|v\|_2^2$
- $V = \mathbb{R}^2 \quad \langle v, w\rangle = v \cdot w \qquad v^T w$

  $(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = \sum_i x_i y_i$
- $V =$ functions $f : \{1, \dots, n\} \to \mathbb{R}$ $\qquad \mathbb{R}^{\{1, \dots, n\}} = \mathbb{R}^n$

  I can define

  $\langle f, g\rangle = \sum_{i=1}^{n} f(i)\, g(i)$ Note: f can be written as $(f(1), f(2), \dots, f(n))$
- "e.g." $V =$ functions $f : [0, 1] \to \mathbb{R}$

  $\langle f, g\rangle = \cancel{\sum_{x \in [0,1]} f(x) g(x)} \qquad \int_0^1 f(x)\, g(x)\, dx$



Thm If V is an inner product space, $\|v\| := \langle v, v\rangle^{1/2}$ is a norm

i.e. every inner product space is a normed space

One would then want to say

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Pf We want to show

$$\|x+y\| \leq \|x\| + \|y\|$$

$$\Updownarrow$$

$$\langle x+y, x+y\rangle \leq \|x\|^2 + 2\|x\|\|y\| + \|y^2\|$$

$$\Updownarrow$$

$$\langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \leq \langle x, x\rangle + 2\|x\|\|y\| + \langle y, y\rangle$$

$$\Updownarrow$$

$$\langle x, y\rangle \leq \|x\|\,\|y\|$$ Cauchy-Schwartz Inequality

If $v, w \in \mathbb{R}^n$, C-S says

$$\frac{\langle v, w\rangle}{\|v\|\|w\|} \leq 1$$

$\|$

$\cos\theta$, where θ is the angle between v and w

This allows us to compute angles in any inner product space e.g.

$V = \mathbb{R}^{[0,1]}$ $\quad f(x) = x \quad g(x) = x^2$

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$\|f\| = \langle f, f\rangle^{1/2} = \left(\int_0^1 [f(x)]^2 dx\right)^{1/2} = \left(\int_0^1 x^2 dx\right)^{1/2} = \left(\frac{1}{3}\right)^{1/2} = \frac{1}{\sqrt{3}}$$

$$x^4 \quad \ldots \quad = \frac{1}{\sqrt{5}}$$

$\|g\| = \ldots\ldots$

$$\text{so}\quad \cos\theta = \frac{\langle f, g\rangle}{\|f\|\|g\|} = \frac{\frac{1}{4}}{\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{5}}\right)} = \frac{\sqrt{15}}{4}$$

$V = \mathbb{R}^{[0,1]}$ $\quad f(x) = x \quad g(x) = 2x \qquad \left(\int_0^1 (2x)^2\,dx\right)^{1/2} = \left(\frac{4}{3}\right)^{1/2}$

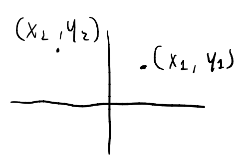
$$\|f\| = \frac{1}{\sqrt{3}} \qquad \|g\| = \frac{2}{\sqrt{3}}$$

$$\cos\theta = \frac{\langle f, g\rangle}{\|f\|\|g\|} = \frac{\frac{2}{3}}{\frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}} = \frac{\frac{2}{3}}{\frac{2}{3}} = 1$$

$\theta = 0$

Suppose x = house price, y = interest rate

(x_i, y_i) are many measurements of these variables

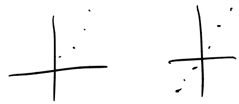


CORRELATION coefficient ρ measures something about the relation between the variables

Fact: the correlation is the angle between

$\vec{x}$ $(x_1, \dots, x_N)$ ← If everything is normalized to the mean 0

$\vec{y}$ $(y_1, \dots, y_N)$



$(1, 2, 3, 4, 5) - (3, 3, 3, 3, 3)$

$(2, 3, 4, 5, 6) - (4, 4, 4, 4, 4)$

$(-2, -1, 0, 1, 2)$

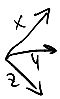
$\rho = 1 \Leftrightarrow y_i = a x_i \ \forall i \ a > 0 \Leftrightarrow \theta = 0$

$\rho > 0$ $\vec{x}, \vec{y}$ are at an acute angle

$\rho < 0$ ... obtuse angle

$\rho = 0$ ... orthogonal / perpendicular

If x is positively correlated with y, and y is positively correlated with z, ...?



<u>Pf</u> of Cauchy-Schwartz " $\langle x, y \rangle \leq \|x\| \|y\|$ "

Let $\lambda \in \mathbb{R}$, and consider

$$\langle x - \lambda y, x - \lambda y \rangle = \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle$$

$$= \|x\|^2 - 2\lambda \langle x, y \rangle + \lambda^2 \|y\|^2$$

Let's think of this as a function of λ

$p(\lambda) = a\lambda^2 + b\lambda + c$

$a = \|y\|^2$

$b = -2\langle x, y \rangle$

$c = \|x\|^2$

$x - \frac{\langle x y \rangle}{\|y\|^2} \cdot y$

By positive definitions,

$p(\lambda) \geq 0$ <u>for all</u> λ

What does $P(\lambda) \geq 0 \;\; \forall \lambda$ say about a, b, c?

Calculus approach:

$P(\lambda)$ is minimized when $P'(\lambda) = 2\lambda a + b = 0$

$$\lambda = -\frac{b}{2a} = \frac{2\langle x, y\rangle}{2\|y\|^2} = \frac{\langle x, y\rangle}{\|y\|^2}$$

Plug in this value:

$$P\left(-\frac{\langle x, y\rangle}{\|y\|^2}\right) = \|x\|^2 - \frac{2\langle x, y\rangle}{\|y\|^2} \cdot \langle x, y\rangle + \frac{\langle x, y\rangle^2}{\|y\|^4}\|y\|^2$$

$$= \|x\|^2 - \frac{2\langle x, y\rangle^2}{\|y\|^2} + \frac{\langle x, y\rangle^2}{\|y\|^2}$$

$$= \|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2} \geq 0$$

$$\|x\|^2 \geq \frac{\langle x, y\rangle^2}{\|y\|^2}$$

$$\frac{\langle x, y\rangle^2}{\|x\|^2\|y\|^2} \leq 1$$

$$\frac{\langle x, y\rangle}{\|x\|\,\|y\|} \leq 1$$

A linear algebra approach:

$a\lambda^2 + b\lambda + c \geq 0 \;\; \forall \lambda \Leftrightarrow b^2 - 4ac \leq 0$

$\Rightarrow$ either

- roots are complex $b^2 - 4ac < 0$
- roots are double $b^2 - 4ac = 0$

Summary: we've proven C-S inequality

$\Rightarrow$ for every IPS, $\|v\| = \langle v, v\rangle^{1/2}$ is a norm

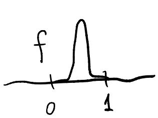
Now, on functions $f: [0,1] \to \mathbb{R}$ we have <u>two</u> norms

$$\|f\|_2 = \left(\int_0^1 f(x)^2\right)^{1/2}$$

$$\|f\|_{sup} = \sup_{x \in [0,1]} |f(x)|$$

$\|f\|_{sup}$ ($= \|f\|_\infty$) is small $\Rightarrow$ f is <u>always</u> small

$\|f\|_2$ is small $\Rightarrow$ f is <u>usually</u> small



$\|f\|_{sup}$ big

$\|f\|_2$ small