MATH 521 Lecture 11

METRIC SPACES

A generalization of the notion of "distance"

Def A <u>metric space</u> is a set X with function $d: X \times X \to \mathbb{R}$

- $d(x,y) \geq 0$ $\quad d(x,y) = 0$ iff $x = y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$

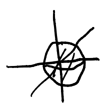
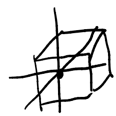
e.g. 

Fact If V is a normed vector space, then the function $d(x,y) = \|x-y\|$ is a metric

Def X metric space w/ metric d $(x \in X)$

The <u>open ball</u> of radius around x is

$B_{X,d}(x,r) = \{ y \in X,\ d(x,y) < r \}$

e.g. $B_{\mathbb{R}^3, d_{Euc}}(0,1) =$  $\quad B_{\mathbb{R}^3, d_{L^\infty}}(0,1) =$ 

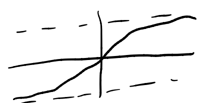
$X =$ bounded functions $f: \mathbb{R} \to \mathbb{R}$

$\|f\|_{sup}$

$d(f,g) = \|f-g\|_{sup}$

$B(0,1) = \{ f : d(f,0) < 1 \} = \{ f : \|f-0\|_{sup} < 1 \} = \{ f : \|f\|_{sup} < 1 \}$

Consider $f(x) = \frac{2}{\pi} \arctan(x)$



$\|f\|_{sup} = 1$ so f is <u>not</u> in $B(0,1)$

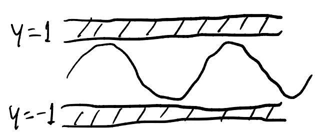
even though $|f(x)| < 1$ for all x

For $f \in B(0,1)$ we need $\|f\|_{sup} < 1$ i.e. $\|f\|_{sup} \leq c$ for some $c < 1$

$\Leftrightarrow$ c is an upper bound for $|f(x)|$ for some $c < 1$

$\forall x$ $|f(x)| < c$ for some $c < 1$

$|f|$ is "bounded away from 1"



TOPOLOGY

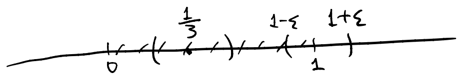
X is a metric space, $S \subset X$

Def The <u>interior</u> of S, denoted $\text{int}(S)$, is the set

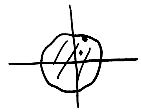
$$\{s \in S : \exists \varepsilon > 0 \ \ B(s, \varepsilon) \subset S\}$$

e.g. $X = \mathbb{R}$ $S = [0,1] = \{x : 0 \le x \le 1\}$

$\text{int}(S) = (0,1)$



$$B(\tfrac{1}{3}, \tfrac{1}{6}) = (\tfrac{1}{6}, \tfrac{1}{2}) \subset S$$
$$\{x : \tfrac{1}{6} < x < \tfrac{1}{2}\}$$

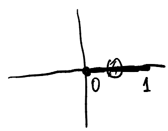


$X = \mathbb{R}^2$ $S = \{x^2 + y^2 \le 1\}$

$\text{int}(S) = \{(x,y) : x^2 + y^2 < 1\} = B(0, 1)$

$\text{int}(\text{int}(S)) = B(0,1)$

Def We say a subset $S \subset X$ is <u>open</u> if $\text{int}(S) = S$



More examples

$X = \mathbb{R}^2$ $S = [0,1]$ on the x-axis

$\text{int}(S) = \emptyset$

$X = \mathbb{R}^1$ $S = \mathbb{Q}$

$\text{int}(S) = \{\frac{a}{b} : \exists \varepsilon > 0 : (\frac{a}{b} - \varepsilon, \frac{a}{b} + \varepsilon) \subset \mathbb{Q}\} = \emptyset$

$X = \mathbb{R}$ $S = Irr$

$\text{int}(Irr) = \emptyset$ (dense)

Thm For any $S \subset X$, $\text{int}(\text{int}(S)) = \text{int}(S)$, $\text{int}(S)$ is open

$\text{int}(\text{int}(S)) \subset \text{int}(S)$

Pf Let $p \in \text{int}(S)$ to prove: $\text{int}(S) \subset \text{int}(\text{int}(S))$

Then there is a r s.t. $B(p, r) \subset S$

We need to show that $\exists r'$ s.t. $B(p, r') \subset \text{int}(S)$

Claim: $r' = \frac{1}{2} r$ will do

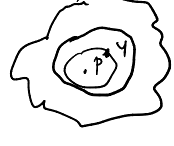
Let y be a point in $B(p, \frac{r}{2})$, I need to show $y \in \text{int}(S)$

To do that, we need to show $\exists$ some $\varepsilon > 0$ s.t. $B(y, \varepsilon) \subset S$

Claim: $\varepsilon = \frac{1}{2} r$ will do

Let $z \in B(y, \frac{r}{2})$. I need to show $z \in S$

$$d(p,z) \le \underbrace{d(p,y)}_{<\frac{r}{2}} + \underbrace{d(y,z)}_{<\frac{r}{2}} < r$$



$\mathbb{Q} \subset \mathbb{R}$

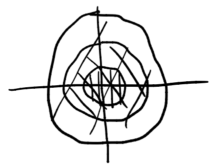
$\mathrm{int}(\mathbb{Q}) = \emptyset$

So $z \in B(p,r) \subset S$

<u>Prop</u> The union of any collection of open sets is open

The intersection of any finite collection of open sets is open

example to worry about $X = \mathbb{R}^2$



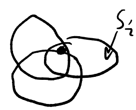
$S = B(0, 2^{-i}) \quad i = 0, 1, 2, 3, \ldots$

$\bigcap_i S_i = \{(0,0)\}$

$\mathrm{int}(\bigcap_i S_i) = \emptyset$

<u>Pf</u> Let ~~$S_1, S_2, S_3, \ldots$~~ $\{S_i\}_{i \in T}$ be a collection of open sets

First, show $\bigcup_{i \in T} S_i$ is open



If $x \in \bigcup_{i \in T} S_i$, $x \in S_i$ for some i

S_i is open, so $\exists r > 0$ s.t. $B(x,r) \subset S_i \subset \bigcup_{i \in T} S_i$ ✓

Now suppose $s \in \bigcap_{i \in T} S_i$ so $s \in S_i$ for <u>all</u> $i \in T$

Let's try to prove $\bigcap_{i \in T} S_i$ is open

For each i, there is a $B(s, r_i) \subset S_i$

I want there to be an r s.t. $B(s,r) \subset \bigcap_{i \in T} S_i$,

i.e. $B(s,r) \subset S_i$ for all i

Choose $r > 0$ to be smaller than any r_i

in fact, might as well make r as large as possible

by taking $r = \inf r_i$

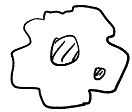
PROBLEM: all the r_i are > 0, but $\inf r_i$ might be 0!

e.g. if $r_i = 1/2^i$

<u>BUT</u> if T is <u>finite</u>, we can just take $r = \min_{i \in T} r_i$

and then $r > 0$ ✓

Prop In fact, every open set $S \subset X$ is a union of some collection of open balls



Pf If S open, each $x \in S$ has a ball $(x, r_x) \subset S$

Consider $\bigcup_{x \in S} B(x, r_x) \subset S$

Every $x \in S$ is contained in $B(x, r_x) \subset \bigcup B(x, r_x)$