MATH 521 Lecture 12 R.T.

Recall $p \in X$, a <u>neighborhood</u> of p is an open set $U \ni p$
(X ↑ metric space)

<u>Def</u> Let E be a subset of metric space X

We say $x \in X$ is an <u>accumulation point</u> (or a <u>limit point</u>)

of E in X ($x \in L_X(E)$)

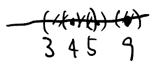
of X if, for every neighborhood $U \ni x$,

$U \cap E$ contains a point not equal to x

Equivalenty: for every $B(x, \varepsilon)$ $\varepsilon > 0$

e.g. $X = \mathbb{R}$

E



4 is an accumulation point

5 is an accumulation point

9 is not an accumulation point (isolated point)

$L_{\mathbb{R}}(E) = [3, 5]$

<u>Prop</u> If $X = \mathbb{R}^n$ with Euclidean metric, and $E \subset X$, then any point in

$\text{int}(E)$ is an accumulation point for E

Let $p \in \text{int}(E)$ Then for some $r > 0$, $B(p, r) \subset E$

Let $U \ni p$ a neighborhood $p \in U = \text{int}(U)$

thus, for some $r' > 0$, $B(p, r') \subset U$ $B(p, \min(r, r')) \subset U \cap E$

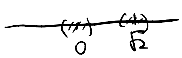
Since $X = \mathbb{R}^n$, $B(p, \varepsilon)$ contains points other than p for all $\varepsilon > 0$,

so we are done

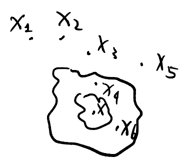
In Hamming Space, $B(111, \frac{1}{2}) = \{111\}$

$L_{\mathbb{R}}(\mathbb{Q}) = \mathbb{R}$

e.g. 2 $X = \mathbb{R}$
$E = \mathbb{Q}$



(every real number x is a limit of some other sequence of rational numbers other than x)



<u>Def</u> Let $x_1, x_2, \ldots$ be a sequence of points in a metric space X

We say $\lim_{i \to \infty} x_i = x$ if, for every neighborhood U of x, $\exists N_U$

such that, for all $i > N_U$, $x_i \in U$

Equivalently: For every $\varepsilon > 0$, $\exists N_\varepsilon$ s.t. for all $i > N_\varepsilon$,

$$x_i \in B(x, \varepsilon) \quad \text{i.e. } d(x, x_i) < \varepsilon$$

Thm Let $E \subset X$. Then x is in $L_X(E)$ iff there exists a sequence $x_1, x_2, \dots$ in $E \setminus x$ with $\lim_{i \to \infty} x_i = x$

Pf Let x be an accumulation point

For each i, $B(x, 2^{-i}) \cap E$

contains a point not equal to x_i

pick and call it x_i

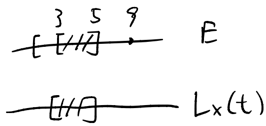
Claim: $\lim_{i \to \infty} x_i = x$

For any $\varepsilon > 0$, take $N = (-\log_2 \varepsilon) + 100$

For all $i > N_\varepsilon$,

$$x_i \in B(x, 2^{-i}) \subset B(x, 2^{-N_\varepsilon}) = B(x, 2^{-100}\varepsilon)$$

So indeed $d(x, x_i) < \varepsilon$



Def We say E is closed in X if $L_X(E) \subset E$

(Why not $= E$? Because of isolated points)

Def We say $E \subset X$ is closed if its complement $\bar{E}$ $(:= X \setminus E)$ is open

e.g. $X = \mathbb{R}$

	open	closed	
$x \le 0$ or $x \ge 1$	X ($x = 0$)	✓	
$\mathbb{R}$	✓	✓	clopen
$0 < x \le 1$	X ($x = 1$)	X ($x = 0$)	
$\mathbb{Q}$	X	X	

Let's prove these two are equivalent

One way: Suppose $\bar{E}$ is open.

Let $p \in \bar{E} = \text{int}(\bar{E})$

so $\exists B(p, \varepsilon) \subset \bar{E}$ $\quad \varepsilon > 0$

$\quad\quad\quad \| $

$\quad\quad\quad U$

Thus U is a neighborhood of p and $U \cap E = \emptyset$

Conclude p is not in $L_X(E)$

To sum up:

If p is not in E, then p is not in $L_X(E)$

$$\Updownarrow$$

if p is in $L_X(E)$, then p is in E

$$\Updownarrow$$

$$L_X(E) \subset E$$

Other way:

Suppose $L_X(E) \subset E$. We want to show $\overline{E}$ is open

Let $p \in \overline{E}$. We need to show there exists some $U \ni p$ with $U \in \overline{E}$

Since $p \notin E$, $p \notin L_X(E)$; so there is some neighborhood $U \ni p$ s.t.

$U \cap E$ contains no points not equal to p

$\Rightarrow U \cap E = \emptyset$ (since $p \notin E$)

$\Rightarrow U \subset \overline{E}$ ✓

Def The <u>closure</u> of E in X is $\text{clos}(E) = E \cup L_E(X)$

Note: $\text{clos}(E) = E$ iff E is closed

Two equal definitions:

- $\text{clos}(E) = \overline{\text{int}(\overline{E})}$
- $\text{clos}(E) =$ intersection of all closed subsets of X containing E

Fact The intersection of any collections of closed sets is closed, the union of any <u>finite</u> collection of closed sets is closed

e.g. Consider the union of $[0, r]$ over all r in $(0, 1)$ is $[0, 1)$

$[0, 1/2]$

$\cup [0, 3/4]$

$\cup [0, \ldots]$