MATH 521 Lecture 13 R.T.

$1, \boxed{\frac{1}{2}}, 1, \boxed{\frac{1}{3}}, 1, \boxed{\frac{1}{4}}, 1, \boxed{\frac{1}{5}}, \ldots$

indecisively wants to converge to both 0 and 1

<u>Def</u> We say x is a <u>subsequential limit</u> of $x_1, x_2, x_3, \ldots$

if $\exists$ a sequence $x_{i_1}, x_{i_2}, x_{i_3}, \ldots$ $\quad i_1 < i_2 < i_3$

with $\lim_{j\to\infty} x_{i_j} = x$

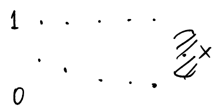
In the example above, we have subsequential limits of 0 and 1

given by $O, \square$

But also: $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots$

<u>Prop</u> If $\lim_{i\to\infty} x_i = x$, then x is the only sequential limit

In fact, $O \ \square \ O \ \square \ \ldots$ has 0 and 1 as its <u>only</u> subsequential limits



<u>Pf</u> of this claim:

Suppose $x \neq 0$ or 1

want to show there is no infinite subsequence of $\underline{x}$ converging to x

It suffices to show: for some $\varepsilon > 0$,

$B(x, \varepsilon)$ contains only finitely many x_i

Idea: Choose $\varepsilon, \varepsilon_1$, and ε_2 s.t. $B(x,\varepsilon)$, $B(1,\varepsilon_1)$, and $B(0,\varepsilon_2)$

are all disjoint

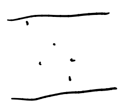
We know that $\exists N_1$ s.t. for all x_i will $i > N_1$, $x_i \in B(1, \varepsilon_1)$
$\exists N_2$ s.t. for all $\boxed{x_i}$ will $i > N_2$, $x_i \in B(1, \varepsilon_2)$

Thus, if $x_i \in B(x, \varepsilon)$, we have $i < \max(N_1, N_2)$

<u>Remark</u>: this can get weirder, e.g.

We showed the rationals were countable, $\in [0,1]$

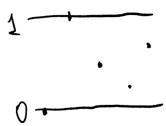
Let $x_1, x_2, x_3, x_4, x_5, \ldots$



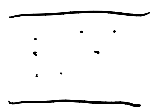
<u>Every</u> real number in $[0,1]$ is a subsequential limit

$1, 2, 3, 4, 5, 6, 7, \ldots$ has no subsequential limit at all

<u>Thm</u> (Bolgano–Weierstress thm) Any <u>bounded</u> sequence of real numbers has a subsequential limit



<u>Thm</u> (Baby Bolgano–Weierstress thm) If K is a <u>finite</u> set of real numbers, then any sequence $x_1, x_2, \ldots$ in K has a subsequential limit



<u>Pf</u> Pigeonhole principle: there will be some $x \in K$ s.t. $x_i = x$ for infinitely many i, which gives a <u>constant</u> subsequence converging to x

SUBS. LIMITS & ACCUMULATION PTS.

<u>Prop</u> Let $x_1, x_2, x_3, \ldots$ a sequence in X

Let $E = \{x_1, x_2, \ldots\} \subset X$

If x is an accumulation pt of E

it is a subsequential limit of x_1, x_2, x_3

Bolgano-Weierstress (CH version)

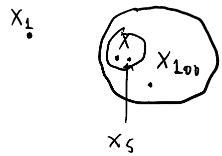
Every infinite subset of $[0,1]$ has an accumulation point

(not true for $\mathbb{R}$ - e.g. $\mathbb{Z}_{>0} = \{1, 2, 3, 4, \ldots\} \in \mathbb{R}$ has no accumulation pts.)

Pf Let's just build a subsequence $x_{i_1}, x_{i_2}, \ldots \to x$

$x_{i_1} = x_1$

Let x_{i_2} be a point of $E \neq x$
in $B\left(x, \frac{d(x, x_1)}{100}\right)$



Let x_{i_3} be a point of $E \neq x$
in $\cancel{B\left(x, \frac{d(x, x_{i_2})}{100}\right)}$

$B\left(x, \min_{j \le i_2} \frac{d(x, x_j)}{100}\right) \Rightarrow i_3 > i_2$

COMPACTNESS

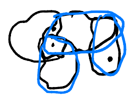
Def Let K be a subset of a metric space X
We say K is compact if, for any collection of open sets of X
$(\{U_s\}_{s \in S})$ which covers K (i.e. $K \subset \bigcup_{s \in S} U_s$)
there is a finite subcollection $U_1, U_2, \ldots, U_N$ which still covers K

Compact Sets

A finite set K is compact

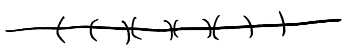
For each $x \in K$, let U_x be an open set in the collection which contains x

Then $\{U_x\}_{x \in K}$ is my collection that covers k



$\mathbb{R}$ is noncompact $\mathbb{Z}$ is noncompact

Consider the collection
$(-2, 0), (-1, 1), (0, 2), (1, 3), (2, 4), \ldots$

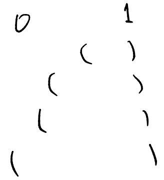


The union of any finite subcollection is bounded and cannot cover all of
$\mathbb{R}$ or all of $\mathbb{Z}$

Indeed, any unbounded subset of $\mathbb{R}$ is noncompact

- $(0,1)$ is noncompact

Pf Consider the collection $(\frac{1}{2}, 1), (\frac{1}{4}, 1), (\frac{1}{8}, 1), (\frac{1}{16}, 1), \ldots$



their union is $(0,1)$

BUT any finite subcollection of these misses x for some (very small) $x > 0$

so does not cover $(0,1)$

Thm (Heine-Borel) $[0,1]$ is compact

Let's try to cover $[0,1]$ with open intervals

$(\frac{1}{2}, 1)$ $\quad$ $(\frac{1}{2}, 1.01)$ $\quad$ $(\frac{1}{2^i}, 1.01)$

$(\frac{1}{4}, 1)$ $\quad$ $(\frac{1}{4}, 1.01)$

$\vdots$ $\quad$ $\vdots$

$(-.0001, 1.01)$

Thm Let $x_1, x_2, \ldots$ be an infinite sequence in a compact subset $K \subset X$

Then $x_1, x_2, \ldots$ has a convergent subsequence where limit is in K

Rmk We express the def of compactness in terms of open subsets of X, but we could have just used open subsets of K

Uses: if U open in X and $E \subset X$, then $U \cap E$ is open in E

e.g. $X = \mathbb{R}$ $\quad E = [0,1]$ $\quad U = (-1, 1)$ $\quad \underset{[0,1)}{U \cap E}$ is open in $\underset{[0,1]}{E}$