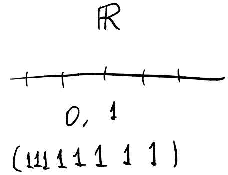
MATH 521 Lecture 14 R.T.

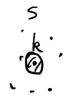
<u>Thm</u> An infinite subset S of a compact set $K \subset X$ has an accumulation point in K



<u>Pf</u> Suppose not.

Then every $k \in K$ is not an accumulation point for S

Thus there is some neighborhood U_k of k such that



- $k \in U_k$
- $U_k \cap S = \{k\}$ or $\emptyset$

<u>EQUIV</u> If $S \subset K$, K compact, $L_K(S) = \emptyset$, then S is finite

$\bigcup_{k \in K} U_k \supset K$, i.e., the $\{U_k\}_{k \in K}$ is a cover of K

BY COMPACTNESS, there is some finite subcollection

$U_{k_1}, U_{k_2}, \ldots, U_{k_n}$ which covers K, whence also S

But each U_{k_i} contains at most one point of S, k_i or none

$S \subset \{k, \ldots, k_n\}$ and in particular S is finite

<u>Cor</u> If $x_1, x_2, x_3, \ldots$ is a sequence of points in K compact then $x_1, x_2, \ldots$ has a subsequential limit in K

e.g. if $K = [a, b]$ this says a bounded sequence of reals has a subsequential limit

<u>Sidebar</u>: In fact, compact sets are always <u>closed</u> so actually <u>all</u> accumulation points / subsequential limits are in K

<u>Pf</u> $S = \{x_1, x_2, x_3, \ldots\}$

Last time we showed that an accumulation point of S is a subsequential limit of $x_1, x_2, \ldots$

If S is infinite, it has an accumulation point in K by <u>Thm</u> and we are done

If S is finite, there is some $s \in S$ which occurs infinitely many times in $x_1, x_2, \dots$

$\Rightarrow$ there is a sequence $s, s, s, s, \dots$

$\Rightarrow$ s is a subsequential limit

More Compactness Facts

<u>Prop</u> A closed subset of a compact K is compact

<u>Sidebar:</u> Recall $[0, 1)$ is open in $[0, 1]$ but not open in $\mathbb{R}$

openness is <u>relative</u>

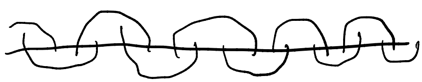
However, if X is a metric space, $Y \subset X$, and $K \subset X$, then K is compact in Y iff K is compact in X

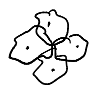
compactness is <u>intrinsic</u>

Recall: we say $K \subset X$ is <u>compact</u> if whenever we have a collection of open sets covering K, some finite subcollection of that collection also covers K

A finite set is compact

$\mathbb{R}$ is not compact

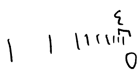




U_1, U_2, U_3, U_4

$[0, 1]$ is compact

$K = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots, 0\}$ is compact



Let $\{U\}$ be a collection of open sets covering K

Then at least one of the open sets in my collection contains 0, call it U_0

U_0 contains 0 and it contains $B_K(0, \varepsilon)$ for some $\varepsilon > 0$

In particular, it contains all $x \in K$ with $|x| < \varepsilon$

This is all but finitely many <u>points of K!</u>

The rest can be covered by one set each.

A collection $U_1, U_2, U_3, \dots$ is a <u>covering</u> of K / <u>covers</u> K means $K \subset \bigcup_i U_i$

Prop A closed subset Y of a compact $K \subset X$ is compact

Pf Suppose we have a covering $\{U_s\}_{s \in S}$ of Y

$U_1, U_2, U_3, \dots$ open sets of K

$U_1, U_2, U_3, \dots, K \setminus Y$ is a covering of K

open in K
because Y is closed

So there is a finite list of these

$U_{i_1}, U_{i_2}, \dots, U_{i_N},$ maybe $(K \setminus Y)$ which still covers K, whence also Y

Then $U_{i_1}, \dots, U_{i_N}$ also cover Y, because $K \setminus Y$ is disjoint from Y

Def X a metric space, $E \subset X$

We say E is <u>bounded</u> if there is some open ball $B(x, r)$ containing E

Prop If $K \subset X$ is compact, then K is bounded "from where"

Pf Choose $x \in X$ $\{B(x,r)\}_{r \in \mathbb{R}_{>0}}$ covers X, whence also K

Since K is compact,
some finite list of them covers K

$$K \subset B(x, r_1) \cup B(x, r_2) \cup \cdots \cup B(x, r_N) = B(x, \max(r_i))$$

$\Rightarrow K$ is bounded

Thm (Heine-Borel) A closed & bounded subset of $\mathbb{R}^n$ is compact
e.g. $[0,1] \subset \mathbb{R}$

Def An <u>n-cell</u> in $\mathbb{R}^n$ is a set $\{a_1 \le x_1 \le b_1, \dots, a_n \le x_n \le b_n\}$

$$= [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

$$= \text{box}$$

We will prove that a box is compact

$x^2+y^2+z^2=1$
closed sphere