MATH 521 Lecture 15

We are proving

<u>Th</u> (Heine-Borel) Every closed subset of $\mathbb{R}^n$ is compact

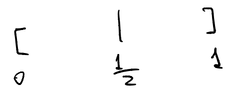
$\uparrow$ this is important

Actually, let's prove a closed $S \subset X$

$[a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact

Actually, we'll prove $[0,1]$ is compact

Method: <u>iterated subdivision</u>



Suppose given a collection $\{U_s\}_{s \in S}$ of open sets covering $[0,1]$.

We want there to be a finite subcollection $U_1, \dots, U_N$ covering $[0,1]$

Q: Is there a finite subcollection covering $[0, \frac{1}{2}]$?

Q: Is there a finite subcollection covering $[\frac{1}{2}, 1]$?

If both answers are YES, we are done

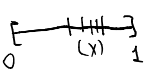
If not, let I_1 be one of these intervals that <u>can't</u> be covered by a subcollection

Now we split I_1 into two halves

if each half can be finitely covered, done

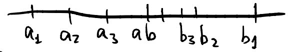
If not, let I_2 be a half, which cannot be covered by a finite subcollection $U_1, \dots, U_N$

We end up with $[0,1] = I_0 \supset I_1 \supset I_2 \supset I_3 \supset \cdots$



I_i cannot be covered by a finite subcollection $U_1, \dots, U_N$

$\text{length}(I_i) = \frac{1}{2^i}$



Prop (Nested Interval Theorem)

If $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$

is a sequence of nested closed intervals in $\mathbb{R}$,

$\exists x \in \mathbb{R}$ s.t. $x \in [a_i, b_i)$ for all i

$[a, a) = \{a\}$

$(a, a) = \emptyset$

Rk false for open intervals, e.g., $(0, 1) \supset (0, \frac{1}{2}) \supset (0, \frac{1}{4}) \supset \cdots$

Pf $\{a_1, a_2, \ldots\}$ is nonempty & bounded above (by b_1) so by LUBP,

$a = \sup\{a_1, \ldots, a_N\}$

$b = \sup\{b_1, \ldots, b_N\}$

By NIT, we have a point x which is contained in I_i for all i

Because $\{U_S\}$ covers X,

there is at least one open set in the collection

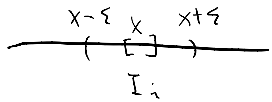
containing x, pick one and call it U_x

Because U_x is open, there is some $\varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset U_x$

But this means that if I take i large enough that $\left(\frac{1}{2^i}\right) < \varepsilon$

then I_i, a closed interval of length $\frac{1}{2^i}$ containing x,

is contained in $(x-\varepsilon, x+\varepsilon) \subset U_x$



So contrary to hypothesis, I_i can be covered

by a finite subcollection $U_1, \ldots, U_N$ with

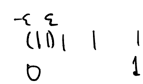
$N = 1$

One way of looking at this,

If $[0, 1]$ is non compact, we can find a "bad spot" x

NIT shows there are no bad spots

How do we prove this is compact!



Given infinite set of U covering X

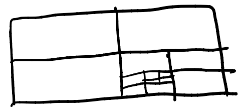
Argument: Because $\{U\}$ covers X,

there is some U in the collection containing 0,

Because U is open, $\exists \varepsilon > 0$ s.t. $(-\varepsilon, \varepsilon) \subset U$

This <u>single</u> U covers all but finitely many of the points of X

What about a box in $\mathbb{R}^n$?



Split box into 2^n (finite) half-sized boxes.

if we can cover each half-sized box with a finite subcollection, we are done

— END OF COMPACTNESS

<u>Cauchy sequences in metric space</u>

<u>Def</u> A sequence $x_1, x_2, x_3, \dots$ in a metric space X is a <u>Cauchy sequence</u> if, for all $\varepsilon > 0$, $\exists N_\varepsilon$ s.t. for all $i, j > N_\varepsilon$ $\quad d(x_i, x_j) < \varepsilon$

Q: Does every Cauchy sequence in X converge to a limit? (in X)

<u>Prop</u> If $x_1, x_2, \dots$ is a Cauchy sequence, it has at most one subsequential limit, $\lim_i x_i = x$

<u>Rk</u> If $X = (0,1)$ $\quad 1/2, 1/4, 1/8, 1/16, \dots$ has no subsequential limit

Why doesn't this have a subsequential limit?

$(\underset{a}{\cdot})$

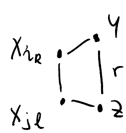
Suppose $a \in (0,1)$ is a subsequential limit, $\quad B(a, \frac{a}{2}) = (\frac{a}{2}, \frac{3a}{2})$

Any subsequence MISSING $\frac{1}{2}$ forever

Suppose y, z are subsequential limits, $y \neq z$

$x_{i_1}, x_{i_2}, x_{i_3}, \dots \to y$

$x_{j_1}, x_{j_2}, x_{j_3}, \dots \to z$



$d(y, z) = r > 0$

There exists some N s.t. for all $i_k > N$, $j_\ell > N$

$$d(x_{i_k}, y) < \frac{r}{100} \qquad d(x_{j_\ell}, z) < \frac{r}{100}$$

$$d(x_{i_k}, x_{j_\ell}) < \frac{r}{100}$$

BUT $r = d(y, z) \leq d(y, x_{i_k}) + d(x_{i_k}, x_{j_\ell}) + d(x_{j_\ell}, z) < \frac{3r}{100}$

Def If every Cauchy sequence in X converges, we say X is <u>complete</u>

(e.g. $\mathbb{R}$ is incomplete)

Prop Every compact metric space is complete

Pf In a compact space, every infinite sequence has a subsequential limit

So every Cauchy sequence has <u>exactly</u> one subsequential limit

which is its limit

Closed sets of $\mathbb{R}$: finite sets, closed interval

$E_0 = [0, 1]$

$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}], [\frac{8}{9}, 1]$

Keep doing this, and define the Cantor set $E := \bigcap_{i=0}^{\infty} E_i$

E is an intersection of closed sets, so it is closed, but

- E has no isolated points
- E contains no closed interval