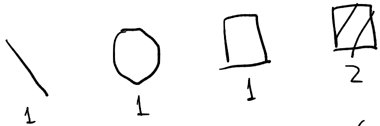
MATH 521 Lecture 16

Dimension

What is the dimension of a set of points?

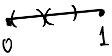


Analyst's version of dimension: (roughly Minkovski dimension)

Let X be a bounded metric space

Define for each $\varepsilon > 0$,

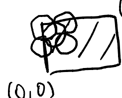
$C(X,\varepsilon)$ = smallest number of ε-balls that can cover X



$X = [0,1]$

$C(X,\varepsilon) \sim \frac{1}{2\varepsilon}$

$\frac{1}{2\varepsilon} + O\left(\frac{1}{2\varepsilon}\right)$

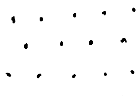


$X = [0,1] \times [0,1]$

$C(X,\varepsilon) \sim c \cdot \varepsilon^{-2}$

$100 \cdot 2\left(\frac{1}{2\varepsilon}\right)^2$

↓ best way



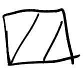
lattice (points as centroid of ε-ball)

$C(X,\varepsilon)$ when X is Hamming space

$\Rightarrow$ error-correcting codes

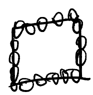
To sum up,

$C(X,\varepsilon) \sim c \cdot \varepsilon^{-1}$



$C(X,\varepsilon) \sim c \cdot \varepsilon^{-2}$

$C(X,\varepsilon) = 3 \cdot \varepsilon^{-0}$



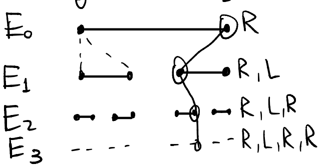
$\sim 4c\varepsilon^{-1}$

$\sim c'' \cdot \varepsilon^{-1}$

Focus on power

Cantor set

Recall



1

$\frac{2}{3}$

$\frac{7}{9}$

$\frac{7}{9}$

$E = \bigcap_{i=0}^{\infty} E_i$

The closet sets we've seen either

- share a closed interval of positive side, or
- countable

<u>Prop</u> The Cantor set is uncountable, closed, and contains no positive length interval
proved last time

Given a length $n+1$ sequence of L, R (e.g. R, L, R, R plotted above)
I got a boundary point in E_n (which is also in E)

<u>Claim 1</u>: If $R, L, R, R, L, R, L, L, \ldots$ is an <u>infinite</u> string of L's and R's,
the esequence $a_n = \{$element of E determined by first $n+1$ terms$\}$
is Cauchy (Key point: $|a_{n+1} - a_n| < \frac{1}{3}n$)
and thus has a limit, which is in E (because E closed)

<u>Claim 2</u>: the map from L-R strings to E is injective
But the set of infinite L-R strings
= set of infinite 0-1 strigs which is <u>uncountable</u>

Why doesn't E contain an interval?
Suppose $[a,b] \subset E$ so $[\frac{c}{3^n}, \frac{c+1}{3^n}] \subset E$

But check: $\frac{c+\frac{1}{2}}{3^n}$ is <u>never</u> in E
(for $c \in \mathbb{Z}_{\geq 0}$)

Choose n s.t. $\frac{1}{3^n} < \frac{b-a}{100}$

What is $C(E, \varepsilon)$?

$$C(E, \tfrac{1}{2} \cdot 1.001) = 1$$
$$C(E, (\tfrac{1}{2} \cdot 1.001) \cdot \tfrac{1}{3}) = 2$$
$$C(E, (\tfrac{1}{2} \cdot 1.001) \cdot \tfrac{1}{9}) = 4$$

$$\left(\frac{1}{3^n}\right)^{-\frac{\log 2}{\log 3}} = 2^n$$

We find $C(E, c \cdot \frac{1}{3^n}) < 2^n$
$$C(E, \varepsilon) < c\,\varepsilon^{-\frac{\log 2}{\log 3}}$$

fra~~ct~~ional dim~~en~~sion