Lecture 17 2022/11/08 Virtual

Functions

To start: let X a metric space,

$E \subset X$ a subset

$f: E \to Y$ and a an accumulation pt in $L_X(E)$

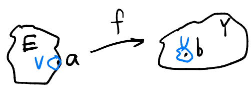
$b \in Y$

Def We say $\lim_{x \to a} f(x) = b$

if: for every neighborhood U of b

$\exists$ a neighborhood V of a s.t.

$f(x) \in U$ for all $x \in V \setminus a$



$f(V) \subset U$

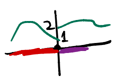
EQUIVALENTLY: For every $\varepsilon > 0$, $\exists\ \delta$ s.t.

If $0 < d(x, a) < \delta$ then $d(f(x), b) < \varepsilon$

in $\mathbb{R}$
$0 < |x - a| < \delta$
$|f(x) - b| < \varepsilon$

Why E? e.g. $X = \mathbb{R}$ we might take $E = (-\infty, 0)$ $f: \mathbb{R}_{<0} \to \mathbb{R}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0^-}$



$E = (-\infty, 0)$: $\lim_{x \to 0} f(x)$ is 1

$E = (0, \infty)$: $\lim_{x \to 0} f(x)$ is 2

$E = \mathbb{R}$: $\lim_{x \to 0} f(x)$ DNE

Def X, Y are metric spaces $f: X \to Y$ a function

We say f is <u>continuous</u> at a point $p \in X$ if, for every neighborhood of $f(p)$,

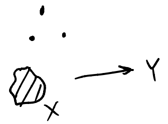
$\exists$ a neighborhood U of p s.t. $f(U) \subset V$

Prop TFAE (the following are equivalent)

- f is continuous at p
- $\lim_{x \to p} f(x) = f(p)$

What if p is not a limit point of X?

Then claim: f is automatically continuous at p



Pf If p not a limit point, $\exists$ some neighborhood U of p s.t. $U \cap X = \{p\}$

In particular, for every neighborhood V of $f(p)$, take $U = \{p\}$: then $f(U) = \{f(p)\} \subset V$ ✓

Def We say $f: X \to Y$ is continuous if it is continuous at every point p of X

Equivalent Def: $f: X \to Y$ is continuous if, for every open set $V \subset Y$, preimage $f^{-1}(V)$ is open in X

$= \{x \in X : f(x) \in V\}$

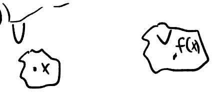
Pf of equivalence:

Let x be a point of $f^{-1}(V)$

I need to show there's an open $U \ni x$ s.t.

$U \subset f^{-1}(V)$ equivalently, $f(U) \subset V$

Preimage $(f^{-1}(V))$ of V = everything that gets mapped to V by S



I know f is continuous at x:

thus, for every neighborhood V of $f(x)$, there is U s.t. $f(U) \subset V$; exactly what we need

Note: $f^{-1}(U^c) = f^{-1}(U)^c$

So if f is continuous $X \to Y$ and C is closed in Y, then $f^{-1}(C)$ is closed in X

Basic facts:

- Constant functions are continuous
- The identity map $i: X \to X$ is continuous
- If $f, g: X \to \mathbb{R}$ are continuous so are $f+g$, $f-g$, fg (though not necessarily f/g)
- Any polynomial $P: \mathbb{R} \to \mathbb{R}$ is continuous
- If $f: X \to Y$ and $g: Y \to Z$ are continuous, so is $g \circ f: X \to Z$
- If $E \subset Y$, and $g: Y \to Z$ is continuous, then $\underset{\uparrow \text{ restricted to}}{g|_E}: E \to Z$ is continuous $g|_E(e) = g(e)$
- A function $X \to \mathbb{R}^n$ $(f_1, f_2, \dots, f_n)$ is continuous if and only if f_i is constant for all i

The extreme value theorem

Recall,

a continuous function on a ~~finite~~ ~~set~~ compact metric space has a maximum

Thm (EVT)
Let K a compact metric space and $f: K \to \mathbb{R}$ a continuous function. Then there exists $x \in K$ s.t. $f(x) = \sup_{p \in K} f(p)$

Rmk Why do economists take real analysis?

Suppose you have S policies we have to make choices about

- marginal tax rate
- \$ schools
- \$ roads
- \$ prisons
- gas tax

Our total "policy bundle" is a point in $\mathbb{R}^S$ which is contained in a space of feasible policy bounds $K \subset \mathbb{R}^S$, which is compact

$f: K \to \mathbb{R}$ is a utility function

$f(p) =$ "How good is policy p"?

Let's also suppose f is continuous

EVT tells you: $\exists p \in K$ s.t. $f(p) = \sup_{p \in K} f(p)$; "there is a best policy"

Pf of EVT

For each $a \in \mathbb{R}$, define

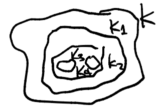
$$K_a = \{x \in K : f(x) \geq a\}$$
$$= f^{-1}([a, \infty))$$

$[a, \infty)$ is closed in $\mathbb{R}$ so K_a is closed in K by continuity

Define $X = \bigcap_{\substack{\text{all } a \text{ s.t.} \\ K_a \text{ is nonempty}}} K_a$

Note that these are <u>nested</u>:

$K_b \subset K_a$ if $b > a$



Claim: X is nonempty

Pf of Claim: Suppose X empty write V_a for $K \setminus K_a$

V_a is open

$$K = X^c = \Big[\bigcap_{\substack{a \text{ s.t.} \\ K_a \text{ nonempty}}} K_a\Big]^c = \bigcup_{\substack{a \text{ s.t.} \\ V_a \neq K}} V_a$$

So these V_a covers K,

so by compactness, some finite collection $V_{a_1}, V_{a_2}, \ldots, V_{a_N}$ of open sets $\neq K$ cover K $\quad a_N = \max a_i$

But $K = \bigcup V_{a_i} = V_{a_N} \neq K$

What we proved: if K compact, every sequence of nested closed nonempty subsets of K has nonempty intersection

Let $x \in X$ Claim: X is a maximum argument: $f(x) = \sup_{p \in K} f(p)$

Suppose not: then $\exists$ some x' s.t. $f(x') > f(x)$

But then $K_{f(x')}$ is nonempty, because it contains $f(x')$

$K_{f(x')}$ does <u>not</u> contain x, because $f(x) < f(x')$

But x is in every nonempty K_a, <u>contradiction</u>