Lecture 18   2022/11/10

Last time: We defined continuity of a function $f: X \to Y$ X, Y metric spaces

analysts definition: $\lim_{x \to p} f(x) = f(p)$

topologists definition: If V is an open subset of Y,
then $\underbrace{f^{-1}(V)}_{= \{x \in X : f(x) \in V\}}$ is an open subset of X

Q: Let X be the space of bounded functions $[0,1] \to \mathbb{R}$

Define $F: X \to \mathbb{R}$ by $F(f) = \|f\|_{sup}$

a function whose input is another function is often called functional

Is F continuous?

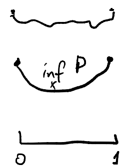
e.g. in Calculus of variations:

$X =$ functions $[0,1] \to \mathbb{R}$

$f(0) = 1$ $f(1) = 1$ "boundry conditions"

arclength$(f) = 1$ 2 $P(f)$

$P: X \to \mathbb{R}$ gravitational potential $\int_0^1 mg\, f(x)\,dx$

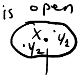


Q: What is $\inf_X P$?

CONNECTEDNESS

Two definitions:

Analysts definition: A metric space X is connected if there is no surjective continuous function $f: X \to \{0,1\}$ ← note $\{0\}$ is open $= B(0, \frac{1}{2})$



$f(x) = 0$

$f(y) = 1$ forall $x \neq 0$

f is continuous,

so $\lim_{i \to \infty} f(y_i) = f(x)$

$\lim_{i\to\infty} f(y_i) = \lim_{i \to \infty} 1 = 1$, $f(x) = 0$

Topologists definition: X is connected if there are no two nonempty open subsets of X U_0, U_1 s.t.

$U_0 \cap U_1 = \phi$

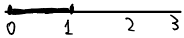
$U_0 \cup U_1 = X$

In this case, $U_1 = U_0^c$ is closed, whence clopen

Why are these equivalent?

If you have continuous, surjective $f: X \to \{0,1\}$ define
$U_0 = f^{-1}(\{0\})$ $U_1 = f^{-1}(\{1\})$

U_i are nonempty (f surjective), open (f continuous)

Note: 

$X = [0,1] \cup [2,3]$ is disconnected

$V_0 = [0,1]$ is open <u>in X</u>

s.t. is the open ball $B_X\left(\frac{1}{2}, \frac{1}{2} + 0.0001\right)$

$V_1 = [2,3]$

Q: Is the Cantor set connected?

No: take $V_0 = e \in E : 0 \leq e \leq \frac{1}{3}$

$-0.0001 < e < \frac{1}{3} + 0.0001$

$V_1 = \ldots \quad \frac{2}{3} \leq e \leq 1$

V_0 is also disconnected

In fact, E is <u>totally disconnected</u>: the only connected subsets of E are single points

Yet E has no isolated points!

$\mathbb{Q}$ is also not connected

$V_0 = \{x \in \mathbb{Q},\ x < \sqrt{2}\}$ $\quad V_1 = \{x \in \mathbb{Q},\ x > \sqrt{2}\}$ covers $\mathbb{Q}$ disjointly

SEQUENCES OF FUNCTIONS

Q: What do we mean when we say a sequence $f_1, f_2, f_3, \ldots$ converges to f?

Q_2: Which properties of the f_i are preserved by f?

One might try the following notion:

Let $f_1, f_2, \ldots$ be functions $X \to Y$

Define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ when this limit exists}$$

If the limit exists for all $x \in X$, we say $f_i \to f$ <u>pointwise</u>

e.g. $X = Y = \mathbb{R}$

f_1

"bump function" supported on $[0,2]$

f_2

$f_2(x) = f_1(x-1)$

f_3

$f_3(x) = f_1(x-2)$

pointwise limit: $f(7) = \lim_{n\to\infty} f_n(7) = 0$

$0, 0, 0, \ldots,$ something, $0, 0, 0, \ldots$

Conclude: $f(n) \to 0$ pointwise

Note that $f_i \to 0$ is definitely <u>not</u> true in the sense of sup norm

That would mean,

for every $\varepsilon > 0$, $\exists N$ s.t. for all $n > N$, $\|f_n - 0\|_{sup} < \varepsilon$

$$\|f_n - 0\|_{sup} = \|f_n\|_{sup}$$

Q: Is the pointwise limit of continuous functions continuous?

Suppose $f_1, f_2, \ldots$ converges to f pointwise

f_n is continuous for all n

We want to show $\lim_{x \to p} f(x) = f(p)$

$$\lim_{x\to p} f(x) = \lim_{x\to p} \lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty}\lim_{x\to p} f_n(x) = \lim_{n\to\infty} f_n(p) = f(p) = f(p)$$

not safe to switch order of limits?

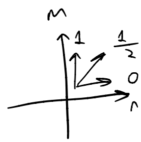
continuity of f_n

pointwise convergence

S can't necessarily switch order of limits

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{m}{m+n} = 0$$

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{m}{m+n} = 1$$



Why sequences of functions?

Taylor series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

e^x is a limit of a sequence of polynomials

could handle functions without Taylor series

Fourier series $\rightarrow$ MATH 522