Def (revisited) A function $f: A \to B$ is

<u>injective</u> if for every $b \in B$, there is at most one $a \in A$ with $f(a) = b$

<u>surjective</u> if for every $b \in B$, there is at least one $a \in A$ with $f(a) = b$

Thus, f is injective <u>and</u> surjective iff for every $b \in B$, there is <u>exactly</u> one $a \in A$ with $f(c) = b$

"A machine you run in reverse"

We call such a f <u>bijective</u> and we denote the unique a with $f(a) = b$ by $f'(b)$

Indeed, when f is a bijection, $A \to B$ we can define a new function $f^{-1}: B \to A$ by $f^{-1}(b) =$ the unique $a \in A$ such that $f(a) = b$

e.g. $f: \mathbb{Q} \to \mathbb{Q}$ $\quad x \mapsto 2x$

If $x \in \mathbb{Q}$, what is $f^{-1}(x)$? $\quad f^{-1}(x) = \frac{1}{2} x$

NOT $\frac{1}{2x}$, which is $[f(x)]^{-1}$

e.g. $\alpha: A \times A^2 \to A^2 \times A \qquad (a, (a', a'')) \mapsto ((a', a''), a'')$

(\mapsto)

FUNCTIONAL COMPOSITION

Given $A \xrightarrow{f} B \xrightarrow{g} C$

I can apply f, then g to get a function $gf : A \to C$ or $g \circ f$
$a \mapsto g(f(a))$ (\circ)

FACTS & EXAMPLES

Not all pairs of fns can be composed. Target of f must match source of g.

$f, g : \mathbb{Q} \to \mathbb{Q}$ $\quad f(x) = x^2$ $\quad g(x) = 2x$

$gf(x) = g(f(x)) = g(x^2) = 2x^2$

$fg(x) = f(g(x)) = f(2x) = 4x^2$

! Function composition is non-commutative

Compare: a $n \times n$ matrix A "is" a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$

AB is a composition of these two compositions

Suppose $f : A \to B$ is a bijection

$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B$ $\quad f^{-1}f : A \to A$

$f^{-1}f(a_0) = f^{-1}(f(a_0))$

the unique a such that $f(a) = f(a_0) = a_0$

i.e. $f^{-1}f = id_A$

$f f^{-1}(b_0) = f\left(\begin{matrix}\text{the unique } a \\ \text{s.t. } f(a) = b_0\end{matrix}\right) = b_0$

$f f^{-1} = id_B$

Given $f: A \to B$, we say $g: B \to A$ is <u>inverse</u> to f

if $fg = id_B$, $gf = id_A$

and in this case we write $g = f^{-1}$

Non-e.g.

C: states $\to$ cities $\quad$ S: cities $\to$ states

C(state) = capital $\quad$ S(city) = state it is in

CS: cities $\to$ cities $\quad$ SC: states $\to$ states

CS(city) = capital of state the city is in $\quad$ $SC = id_{states}$

ORDERED SETS

S is a set

What does it mean to put a set in order?

X ~~To write all elements from smallest to largest~~

For us, it means a way of saying, given $x, y \in S$, which one is larger

Given sets A, B

a <u>relation</u> from A to B is a subset $R \subset A \times B$

An <u>ordering</u> is a certain kind of relation, which we denote $<$

<u>Def</u> An ordering on S is a relation on $S \times S$ such that

- (comparability) $\forall x, y \in S$ <u>exactly</u> one of the following is true:

  $x < y$, $y < x$, $x = y$

- (transitivity) $S = \{$frog, chicken, cow$\}$

  cow > chicken $\quad$ chicken > frog $\quad$ <u>cow > frog</u>

  If $x < y$ and $y < z$, then $x < z$

Notation: An ordered set is a per $(S, <)$ into $<$ is an ordering on S

$x > y$ means $y < x$

$x \leq y$ means $y < x$ or $y = x$

FACT: In an ordered set S, if $x \leq y$ and $y \leq x$, then $x = y$

Proof: Given x, y with $x \leq y$ and $y \leq x$

either $x = y$ or $x < y$

Case 1: $x = y$ DONE

Case 2: $x < y$ Since $y \leq x$, either $y = x$ or $y < x$

Case 2.1: $y = x$ DONE

Case 2.2: $x < y$ & $y < x$ ruled by axiom 1

EXAMPLES

- frog < chicken < cow
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{R}$
- Any subset of an ordered set
- $\mathbb{Z}_{\geq 0} \cup \{\omega\}$ with the ordering $<_\omega$ defined as follows:

$\{0, 1, 2, 3, \ldots\} \cup \{\omega\}$ $\qquad$ $0, 1, 2, 3, \ldots, \omega$ $\qquad$ $-\omega, \ldots, -1, 0, 1, \ldots, \omega$

- If $m, n \in \mathbb{Z}_{>0}$ $m <_\omega n$ iff $m < n$
- $\omega >_\omega n$ for all $n \in \mathbb{Z}_{>0}$

Check axiom 1

given x, y prove exactly one of $x < y$, $y < x$, $y = x$ true

Case 1: $x, y \in \mathbb{Z}_{>0}$, follows from facts about integers

Case 2: $x = \omega$, $y \in \mathbb{Z}_{>0}$ $\quad y < x$

Case 3: $x \in \mathbb{Z}_{>0}$ $y = \omega$ $\quad x < y$

Case 4: $x, y \in \omega$ $\quad x = y$

UPPER BOUNDS

Let S be an ordered set $E \subset S$

We say E is bounded above in S if $\exists s \in S$

st. $\forall e \in E$, $s \geq e$

We say s is an upper bound for E

- $S =$ frog < chicken < cow

  $E = \{$frog, cow$\}$, cow is an upper bound
- $S = \mathbb{Z}_{>0}$ $E = \{1, 2, 4, 7\}$ 1,000,000; 14; 10; 7
- $S = \mathbb{Z}_{>0}$ $E = \mathbb{Z}_{>0}$ Not bounded above
- $S = \mathbb{Z}_{>0} \cup \omega$ $E = \mathbb{Z}_{>0}$ ω is an upper bound
- $S = \mathbb{Q}$ $E = \{x \in \mathbb{Q}, x < 0\}$ 5 is an UB, $\frac{1}{2}$ is an UB.

  0 is an UB

Def We say s is a least upper bound for E if

- s is an upper bound for E
- If t is an upper bound for E, then $t \geq s$