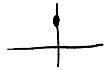
MATH 521 Lecture 20

Last time, we observed that the sequence of narrowing hats

$$f_n = \chi_{[-\frac{1}{n}, \frac{1}{n}]}$$

- does not converge uniformly
- but converges in L^2 to $\chi_{\{0\}}$  <u>and</u> to 0

Solution: <u>declare</u> these to be the same

$$f \sim g \quad \text{if} \quad \|f-g\|_2 = 0$$

<u>Fact</u>: the set of equivalence classes for this relations forms a normed vector space called the space of L^2 functions

e.g. class containing f = class f_1
+ class containing g = class g_1
= class containing $f+g$ = class $f_1 + g_1$

For this to work, we need $f \sim f_1,\ g \sim g_1 \Rightarrow f+g \sim f_1 + g_1$

If $f_1, f_2, \dots$ converges in L^2 to f, does it converge pointwise to f?

i.e., is $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x$

$f(x)$ ← not well defined!

$f(0) = ?$

<u>Thm</u> Let X be a set, Y a complete metric space

$B(X)$ is the normed vector space of bounded functions $X \to Y$

Then $B(X)$ is complete

<u>Pf</u> To prove: if $f_1, f_2, \dots$ is a Cauchy sequence in $B(X)$, then $f_1, f_2, \dots$ converges uniformly to a limit f

For every ε, $\exists N_\varepsilon$ s.t. $d_{B(X)}(f_i, f_j) < \varepsilon$

$$\Leftrightarrow \|f_i - f_j\|_{sup} < \varepsilon$$

$$\Leftrightarrow \sup_{x\in X} |f_i(x) - f_j(x)| < \varepsilon \qquad \sup_{x\in X} d_Y(f_i(x), f_j(x))$$

$$\Leftrightarrow \forall x \in X,\ |f_i(x) - f_j(x)| < \varepsilon$$

e.g. X potato $\Rightarrow \underset{\in Y}{f_1(\text{potato})}, f_2(\text{potato}), \dots$ is a Cauchy sequence in Y

Because Y is complete, has a limit in Y, call it $f(\text{potato})$

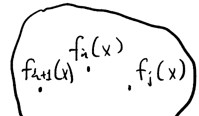
We have thus defined a function $f: X \to Y$ by

$\forall x \quad f(x) = \lim_{n\to\infty} f_n(x)$

i.e., $f_1, f_2, \ldots$ has a pointwise limit f

Remains to show: $f_1, f_2, \ldots$ converges uniformly to f

Choose $\varepsilon > 0$ For all $i > N_\varepsilon$,



we have $f_j(x) \in B(f_i(x), \varepsilon)$

for all $j > i$, for all $x \in X$

Since $f(x) = \lim_{k\to\infty} f_{i+k}(x) \quad d(f(x), f_i(x)) \leq \varepsilon < 1.001\,\varepsilon$

i.e., for all $i > N_\varepsilon$, $\|f - f_i\|_{sup} < 1.001\,\varepsilon$

A complete normed vector space is called a Banach space

A complete inner product space is called a Hilbert space

Thm (Uniform limit theorem) Let $f_1, f_2, \ldots$ be a sequence of continuous functions uniformly converging to f. Then f is continuous

Rmk ① As a consequence, if $B^c(X) \subset B(X)$ is the subspace of continuous functions, then B^c is closed and complete

As a consequence, if $B^{cts}(X) \subset$ bounded $B(X)$ is the subspace of bounded continuous functions, then B^{cts} is closed and complete

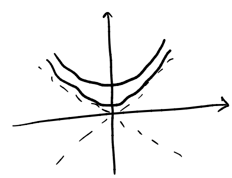
$f_1, f_2, \ldots$ is a Cauchy sequence of cts bounded functions (continuous)

$f_1, f_2, \ldots$ uniformly converges to some f in $B(X)$ because $B(X)$ is complete

By Uniform limit theorem, f is continuous, $f \in B^{cts}(X)$

② Consider the sequence $f_1, f_2, f_3, \ldots$

$f_n = \sqrt{x^2 + \frac{1}{n}}$



$y^2 - x^2 = \frac{1}{n}$

$f_1, f_2, \ldots$ converges uniformly to $f(x) = |x|$

Each f_i is differentiable, but f is not

Actually, let's check uniform convergence

I need to bound $\|f_n - f\| = \sup_x \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}$

To bound this, note $\left(\sqrt{x^2+\frac{1}{n}} - \sqrt{x^2}\right)\left(\sqrt{x^2+\frac{1}{n}} + \sqrt{x^2}\right) = x^2 + \frac{1}{n} - x^2 = \frac{1}{n}$

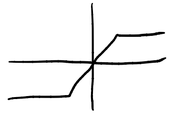
$\sqrt{x^2+\frac{1}{n}} - \sqrt{x^2} = \dfrac{\left(\frac{1}{n}\right)}{\sqrt{x^2+\frac{1}{n}} + \sqrt{x^2}} \leq \dfrac{1}{2n|x|} \longleftarrow x \to 0$ bound cracks $\leq \dfrac{1}{\sqrt{n}}$

and $\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2} \geq 2|x| \geq \sqrt{\frac{1}{n}} \geq \max\left(2|x|, \sqrt{\frac{1}{n}}\right)$

What's going on?

Consider $f_1', f_2', f_3', \dots, f_{100}'$

These just do <u>not</u> converge uniformly



<u>Pf</u> of ULT

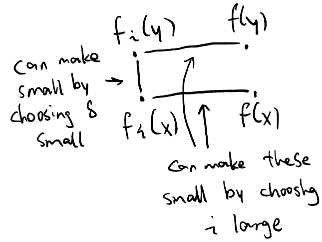
I have $f_1, f_2, \dots$ converging uniformly to f, f_i continuous for all i, want to prove f continuous

For all x, we need to prove f continuous at x

For every $\varepsilon > 0$, $\exists \delta$ s.t. if $d(x, y) < \delta$, $|f(x) - f(y)| \leq \varepsilon$

Choose $N_{\varepsilon/3}$ s.t. for all $x \in X$, all $i > N_{\varepsilon/3}$

$$|f_i(x) - f(x)| < \frac{\varepsilon}{3} \quad \text{(uniform convergence)}$$



Choose some $i > N_{\varepsilon/3}$

Choose δ s.t. for all y s.t. $d(y, x) < \delta$

$$|f_i(x) - f_i(y)| < \frac{\varepsilon}{3} \quad \text{(continuity of } f_i\text{)}$$

Now, with these choices,

$$|f(y) - f(x)| \leq \underbrace{|f(x) - f_i(x)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_i(x) - f_i(y)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_i(y) - f(y)|}_{< \frac{\varepsilon}{3}}$$

$$< \varepsilon$$

What if I didn't have uniformity? Then, there would be

$N_{\varepsilon/3, x}$ s.t. $|f_i(x) - f(x)| < \frac{\varepsilon}{3}$

$N_{\varepsilon/3, y}$ s.t. $|f_i(y) - f(y)| < \frac{\varepsilon}{3}$

Change:

For every $\varepsilon > 0$, $\exists \delta$ s.t. for all y with $d(x, y) < \delta$,

$|f(x) - f(y)| \leq \varepsilon$

Choose $\delta \rightarrow$ Choose δ_i