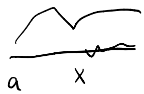
Lecture 21 Virtual

Let $f:(a,b)\to\mathbb{R}$

We write $f(x+)=q$ $\quad x\in(a,b)$ to mean:

$\lim_{n\to\infty} f(x_n)=q$ for every sequence $x_1, x_2, x_3, \ldots \in (x,b)$ converging to x — $\in$: drawn from

"limit as f goes to x from above"



$f(x-)=q$ means something but for sequences in (a,x)

Prop f is continuous at x iff $f(x+)$ exists, $f(x-)$ exists, and $f(x-)=f(x)=f(x+)$

Pf If f is continuous at x, then, for any sequence $x_1,\ldots,x_n$ in (a,b) converging to x $\quad \lim_{n\to\infty} f(x_n)=f(x)$

In particular, this applies to the sequences used to define $f(x+)$ and $f(x-)$, so $f(x+)=f(x-)=x$

Now suppose

and let $x_1, x_2, \ldots$ be a sequence in (a,b) converging to x

I need to show you that $\lim_{n\to\infty} f(x_n)=f(x)$

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, \ldots$

blue: in (a,x)

red: in (x,b)

By $f(x+)=x$, the subsequence of red elements $f(x_i)$ converges to x

$f(x-)=x$, $\ldots$ blue $\ldots$ $f(x_i)$ $\ldots$

So $f(x_1), f(x_2), f(x_3), \ldots$ can be partitioned into three disjoint subsequences, each of which converges to $f(x)$

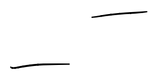
Lemma. If a sequence can be partitioned into a finite union of subsequences, each converging to L, then the sequences converges to L

$$0, 1, 0, \tfrac{1}{2}, 0, \tfrac{1}{3}, 0, \ldots$$

So $f(x_1), f(x_2), \ldots \to f(x)$

Types of discontinuities

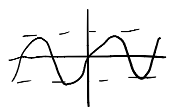
removable: $f(x-) = f(x+)$
but $f(x)$ not equal to these

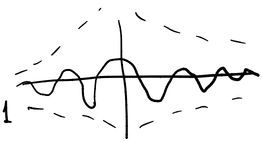


jump: $f(x-)$ and $f(x+)$
both exist but are not equal

essential: Either $f(x-)$ or $f(x+)$ doesn't exist

$$\operatorname{sinc}(x) := \frac{\sin(x)}{x} \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$





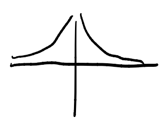
Choices:

$\operatorname{sinc}(x) : \mathbb{R} - 0 \to \mathbb{R}$ continuous

or: $\operatorname{sinc}(x) : \mathbb{R} \to \mathbb{R}$ $\qquad \operatorname{sinc}(0+) = \operatorname{sinc}(0-) = 1$

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$ continuous everywhere

0: removable discontinuity at $x=0$



$f(x) = \frac{1}{x^2}$

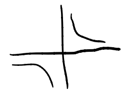
$f(0+)$ DNE $\qquad f(0-)$ DNE

but note: if we consider instead functions from $\mathbb{R}$ to $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$

$f(0+) = f(0-) = \infty$

So we can make f continuous by setting $f(0) = \infty$

$f(x) = \frac{1}{x}$

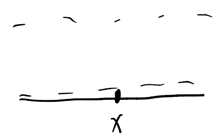


In $\mathbb{R}$, $f(0-) = -\infty$ $\quad f(0+) = \infty$ jump discontinuously

Characteristic function $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$

$\chi_{\mathbb{Q}}(0+) = \text{DNE}$

$\chi_{\mathbb{Q}}(0-) = \text{DNE}$



If $f: \mathbb{R} \to \mathbb{R}$ is a function,

let $\text{Discont}(f) \subset \mathbb{R}$ be the set

$\{x \in \mathbb{R} : f \text{ is discontinuous at } x\}$

$\text{Discont}(\chi_{\mathbb{Q}}) = \mathbb{R}$

$\text{Discont}\left(\begin{smallmatrix}\text{sinc} \\ \text{with} \\ \text{sinc}(0) = 1\end{smallmatrix}\right) = \emptyset$

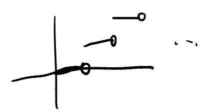
$\text{Discont}(\underbrace{\_\,\dot{}\,\_}_{\chi_{\{0\}}}) = \{0\}$

Can $\text{Discont}(f)$ be <u>any</u> subset of $\mathbb{R}$?

No (More in a bit)

It can be more complicated e.g.

$f(x) = \lfloor x \rfloor$



$\text{Discont}(f) = \mathbb{Z}$

We can have $\text{Discont}(f) =$ Cantor set!

<u>Thm</u> (frota) A set $S \subset \mathbb{R}$ can be $\text{Discont}(f)$ for some $f: \mathbb{R} \to \mathbb{R}$ if and only if it is the union of countbly many closed subsets (any closed)

e.g. this says there is a function $f: \mathbb{R} \to \mathbb{R}$ which is discontinuous at all rationals continuous at all irrationals $\quad \text{Discont}(f) = \mathbb{Q}$

For example: define f by

$f(x) = 0$ if x irrational

$f\left(\frac{p}{q}\right) = \frac{1}{q}$ if $\frac{p}{q}$ is a function in lowest terms

$f(0.5) = \frac{1}{2}$

Suppose $x_1, x_2, x_3, \ldots \to x$ with x irrational

We need to show

$$f(x_1), f(x_2), f(x_3), \ldots \to f(x) = 0$$

irrational x_i $\quad f(x_i) = 0$

rational x_i $\quad f(x_i) = \frac{1}{q_i}$ $\quad x_i = \frac{p_i}{q_i}$

To show that $\frac{1}{q_i} \to 0$, it suffices to show that

$q_i \to \infty$ in $\overline{\mathbb{R}}$ (Exercise)

That is, we have to show that,

for any $E > 0$, $\exists N_E$ s.t. $q_i > E$ for all $i > N_E$

For this, it suffices to show that,

for any $\varepsilon > 0$, and any $x_i \in \mathbb{Q}$ with ε of x,

$q_i > N_\varepsilon$

all rational #s will denominators < 1000

$[\ \ldots \to \sqrt{2}\ ]$

When x is rational, let $x_1, x_2, \ldots$ be a sequence of irrationals converging to x. Then

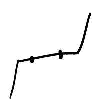
$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$ but $f(x) \neq 0$

Monotone Functions

Def A function $f : (a,b) \to \mathbb{R}$ is monotone nondecreasing if $f(y) \geq f(x)$ whenever $y \geq x$ and strictly increasing if $f(y) > f(x)$ whenever $y > x$

Prop If f is monotone nondecreasing, then

$$f(x-) = \sup_{y<x} f(y) \qquad f(x+) = \inf_{y>x} f(y)$$



monotone ✓

strictly increasing ✗

Idea: if $y_1, y_2, y_3, \ldots$ converges to x from below, and this sequence is nondecreasing, then

$$\lim_{n\to\infty} y_n = \sup_n y_n$$ (monotone convergence thm for sequences)

Note that $f(y) \le f(x) \quad \forall y < x$

So $\sup_{y<x} f(y) \le f(x)$

$= f(x-) \le f(x)$

Similarly, $f(x) \le f(x+)$ $(*)$

$f(x-)$ and $f(x+)$ exist, so no essential discontinuities

If $f(x-) = f(x+)$, then $f(x) =$ both, so f is continuous at x;

no removable discontinuities

$\Rightarrow$ all discontinuities are jump discontinuities

Note also: if $x_1 < x_2$, then $f(x_1+) \le f(x_2-)$

Because if x_3 is in (x_1, x_2), $f(x_1+) \le f(x_3) \le f(x_2-)$

<u>Thm</u> If f monotone nondecreasing, Discont (f) is countable

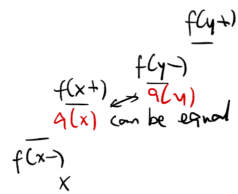
<u>Pf</u> Let $x \in$ Discont (f). Then $f(x-) < f(x+)$

Choose some rational number $q(x)$ with

$$f(x-) < q(x) < f(x+)$$

This defines a function $q:$ Discont $(f) \to \mathbb{Q}$

Claim: q is injective



Why? $q(x) < f(x+) \le f(y-) < q(y)$

$\Rightarrow q(x) < q(y)$

In fact, the set of jump / removable discontinuities of a (not necessarily monotone) $f: \mathbb{R} \to \mathbb{R}$ is countable