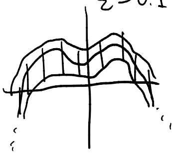
Lecture 22

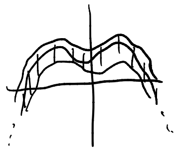
Uniform convergence: A picture

$\varepsilon = 0.1$



For n large enough, f_n is contained within this narrow band

$\varepsilon = 0.05$



For n even larger, f_n is contained in this even narrower band

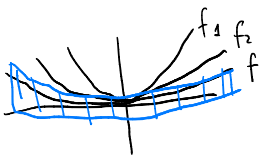
f_n contained in ε-band around f

$\Leftrightarrow$

$f_n(x) \in (f(x) - \varepsilon, f(x) + \varepsilon) \ \forall x$

$|f_n(x) - f(x)| < \varepsilon \quad \forall x$

Pictures of non-uniform convergence



There is <u>no</u> ε and <u>no</u> n s.t.

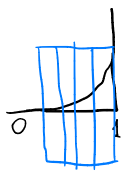
f_n is contained in an

ε-band converged to f

$|f_n(x) - f(x)|$ is unbounded in x

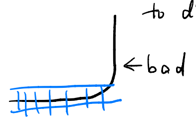
$f_n(x) = x^n \qquad [0,1] \to \mathbb{R}$

$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$



$\varepsilon = 1$

convergent fns could converge to disconvergent fns



$\varepsilon = 0.2$

lim sup & lim inf

Let $s_1, s_2, \ldots$ a sequence of real numbers

Define $\displaystyle \limsup_{n\to\infty} s_n = \lim_{n\to\infty} \overbrace{\sup_{k \geq n} s_k}^{s'_n}$

$\displaystyle \liminf_{n\to\infty} s_n = \lim_{n\to\infty} \underbrace{\inf_{k \geq n} s_k}_{s''_n}$

} (in $\overline{\mathbb{R}}$, the extended reals)

i	1	2	3	4	5	6	7	8	
s_n	1,	-1,	2,	$-\frac{1}{2}$,	3,	$-\frac{1}{4}$,	4,	$-\frac{1}{8}$,	…
s'_n	∞,	∞,	∞,	∞,	∞,	∞,	∞,	∞,	…
s''_n	-1,	-1,	$-\frac{1}{2}$,	$-\frac{1}{2}$,	$-\frac{1}{4}$,	$-\frac{1}{4}$,	$-\frac{1}{8}$,	$-\frac{1}{8}$,	…

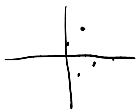
$\limsup s_n = \lim s'_n = \infty$

$\liminf s_n = \lim s''_n = 0$

Prop lim sup and lim inf <u>always</u> exist!
(in $\overline{\mathbb{R}}$; they exist in $\mathbb{R}$ iff sequence is bounded)

Pf s_n' is nonincreasing $\quad$ s_n'' is nondecreasing

So both converge, by the monotone convergence theorem
(extended to $\overline{\mathbb{R}}$ - every nondecreasing or nonincreasing sequence has a limit in $\overline{\mathbb{R}}$)



I think of lim inf as an "eventual infimum"
"The smallest x such that the sequence is $\leq x$ infinitely many times"

Recall $\lim s_n = \infty$ is <u>not</u> the same thing as "s_n unbounded above"

But

Prop $\limsup s_n = \infty$ (inf) ($-\infty$) $\iff$ s_n unbounded above (below)

(Indeed, if s_n is unbounded, so is every tail of s_n, so $s_n' = \infty$ for every n)

Prop lim sup (inf) is the supremum (infimum) of all subsequential limits

In particular, if $s_n \to s = \lim s_n$,
then $\limsup s_n = \liminf s_n = \lim s_n$

and conversely, if $\limsup s_n = \liminf s_n$, then
$\lim s_n$ exists and equals these

INFINITE SERIES

What do we mean by $\sum_{i=1}^{\infty} \frac{1}{i^2}$?

We can define <u>partial sums</u> $s_n = \sum_{i=1}^{n} s$

Define $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$ $\qquad$ a <u>series</u> has a <u>sequence</u> of partial sums

if this limit exists, in which case we say, the infinite series $\sum_{i=1}^{\infty} a_i$ <u>converges</u>

<u>Remark</u>: for this definition to make sure we need to be summing something with an <u>ordering</u>

e.g. can't say "which is the sum of all rational numbers"

<u>Remark</u>: $n > m$ $\quad S_n - S_m = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i = \sum_{i=m+1}^{n} a_i$

Suppose S_n converges. Then, for every $\varepsilon > 0$, $\exists N$ s.t.

$|S_n - S_m| < \varepsilon$ for all $n, m > N$

In particular, for all $m > N$,

$|S_{m+1} - S_m| < \varepsilon$

$= |a_{m+1}|$

Conclude: if $\sum a_i$ converges, $\lim a_i$ exists and $= 0$

(but converse does not hold!)

<u>Example</u> (geometric series)

$x \in \mathbb{R}$ $\quad \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \cdots$

$S_n = \sum_{i=1}^{n} x^n = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$

$(1 + \cancel{x} + \cdots + \cancel{x^n})(1 - x) = 1 - x^{n+1}$

$- x - x^2 - \cdots - x^n - x^{n+1}$

If $|x| \geq 1$,

x^i does not converge to 0

So series does not converge

when $|x| < 1$, this converges to $\frac{1}{1-x}$

Should we say

$1 + 2 + 4 + 8 + \cdots = \frac{1}{1-2} = -1$? (not in $\mathbb{R}$, but yes in "2-adic numbers")

$x = 1$ $\quad 1 + 1 + 1 + 1 + \cdots$ just shouldn't exist

$x = -1$ $\quad 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ (Grandi series)

$(1 - 1) + (1 - 1) + (1 - 1) \to 0$?

partial sums $\quad 1, 0, 1, 0, 1, 0, \ldots$

$s - 1 = -s$

$2s - 1 = 0$

$s = \frac{1}{2}$?