Lecture 24

Sums of Series of Functions

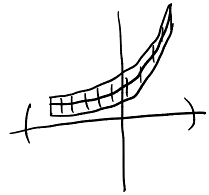
$f_1, f_2, f_3, \ldots$ a sequence of functions

$$\sum_{i=1}^{\infty} f_i = \lim_{n\to\infty} S_n$$

w/r/t whenever notion of convergence we have in mind

$$S_n = \sum_{i=1}^{n} f_n$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = e^x$$



e.g. $f_n : (-1, 1) \to \mathbb{R}$

$f_n(x) = x^n$

$1 + x + x^2 + x^3 + x^4 + \cdots$

By what we clearly did,

$S_n(x) = \frac{1-x^n}{1-x}$ converges to $\frac{1}{1-x}$ pointwise

BUT this is not uniform

BUT: If we restrict domain to

$[-(1-\delta), 1-\delta]$ $\delta > 0$,

convergence is uniform

How big can S_n be?

$S_n(x) = 1 + x + x^2 + \cdots + x^n$

each summand has 1 1 at most 1, so

$|S_n(x)| \leq 1 + 1 + \cdots + 1$

$= n + 1$

This follows from

<u>THM</u> (Weierstrass M-test)

Let $M_1, M_2, \ldots$ a sequence of non-negative reals

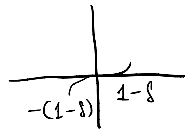
s.t. $\sum_{i=1}^{\infty} M_i$ converges

$f_1, f_2, \ldots$ a sequence of functions $X \to \mathbb{R}$

s.t. $|f_i(x)| \leq M_i \quad \forall x \in X$

Then $\sum_{i=0}^{\infty} f_i$ converges uniformly

Why does this apply to the geometric series?



I need M_i

s.t. $|f_i(x)| \le M_i \quad \forall x \in [-(1-\delta), 1-\delta]$

$$|x^i| \le M_i \quad \forall x \in [-(1-\delta), 1-\delta]$$

$$M_i = (1-\delta)^i$$

If I tried to do this with $x = (-1, 1)$

Vista Fourier Series

metal bar

$T(x,t) =$ temperature of the bar at position x and time t

Given $T(x,0)$, what is $T(x, z)$?

The answer is given by the heat equation which expresses

$$\frac{\partial^2}{\partial t^2} T(x,t) \text{ in terms of } T(x,t)$$

For some special choices of $T(x,0)$

this equation is not hard to solve

$$T(x,0) = c \quad \Rightarrow T(x,t) = c$$

linear homogeneous differential equation

$$T(x,0) = \sin(x) \qquad T(x,t) = e^{-t}\sin(x)$$

$$T(x,0) = \sin(nx) \qquad T(x,t) = e^{-n^2 t}\sin(nx)$$

$$7 + 5\sin(x) + 2\sin(2x) + 8\sin(3x) + \cdots$$

Q: Which functions can be expressed as infinite sums of trignometric functions? (All functions??)

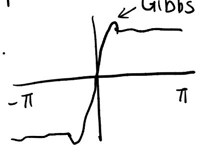
$$\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$$

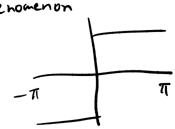
Fourier series

Not obvious that this converges even pointwise

Gibbs phenomenon

$$T(x,t) = e^{-t}\sin(x) + \frac{1}{3}e^{-9t}\sin(3x) + \frac{1}{5}e^{-25t}\sin(5x) + \cdots$$





Differentiation

$f : X \to \mathbb{R}$ $\quad X \subset \mathbb{R}$ $\quad X = [0,1]$ or (a,b)

Def We say f is <u>differentiable</u> at $x \in X$ if for all sequences $t_1, t_2, \ldots \to x$

$$\lim_{t \to x} \frac{f(t)-f(x)}{t-x} \text{ exists}$$

in this case, the limit is always the same and we call it $f'(x)$

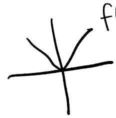
Note: if this limit exists,

$$\lim_{t \to x} f(t) - f(x) = 0, \text{ so}$$

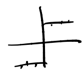
f differentiable at $x \Rightarrow f$ continuous at x

but not $\Leftarrow$

$f(x) = |x|$



not differentiable at 0; $\dfrac{f(t)-f(0)}{t-0} = \dfrac{|t|}{t}$



e.g. let's calculate $(fg)'$

$$\lim_{t \to x} \frac{f(t)g(t) - f(x)g(x)}{t-x}$$

$$= \lim_{t \to x} \frac{f(t)g(t) - f(t)g(x) + f(t)g(x) - f(x)g(x)}{t-x}$$

$$= \lim \frac{f(t)(g(t)-g(x))}{t-x} + \lim \frac{(f(t)-f(x))g(x)}{t-x}$$

$$= \lim_{t \to x} f(t) \lim \frac{g(t)-g(x)}{t-x} + \lim \frac{f(t)-f(x)}{t-x} \lim g(x)$$

$$= f(x)g'(x) + f'(x)g(x)$$

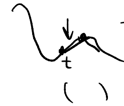
Prop Let $f : (a,b) \to \mathbb{R}$ and let x be an <u>maximum</u>, i.e.

$$f(x) \geq f(y) \quad \forall y \in (a,b)$$

and suppose f is differentiable at x

Then $f'(x) = 0$

$g(t)$ is slope of this <u>chord</u>



Pf Define $g : (a,b) \to \mathbb{R}$

$$g(t) = \frac{f(t) - f(x)}{t - x}$$

When $t < x$, $f(t) \leq f(x)$ $z<x$ so $g(t) \geq 0$

When $t > x$, $f(t) \leq f(x)$ $z>x$ so $g(t) \leq 0$

Differentiability tells us that

$\lim_{t \to x} g(t)$ exists, for <u>every</u> sequence $z_1, z_2, \ldots \to x$

In particular, every sequence converging to x from below has

$g(t_1), g(t_2), \ldots$ are non-negative, so

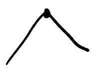
$\lim_{t \to x} g(t) = f'(x)$ ... above

$\Rightarrow g(x^-) = f'(x) \geq 0$

$g(x^+) = f'(x) \leq 0$

$= 0$

Without assumption of differentiability, this doesn't work,

f  g 

<u>Lemma</u> If f is continuous on $[a,b]$ and $f(a) = f(b)$, then: f has an extremum in (a,b)

<u>Pf</u> $[a,b]$ is compact

x be maximum, y minimum

$$f(x) = \max_{t \in [a,b]} f(t)$$

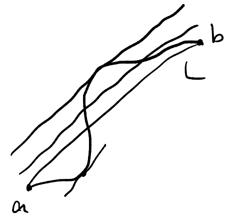
$$f(y) = \min_{t \in [a,b]} f(t)$$

If x or y is in (a,b), ✓

If not, $f(x) = f(y)$, so f is constant ✓

MEAN VALUE THM

$f : [a, b] \to \mathbb{R}$ continuous, differentiable on (a, b)



Thm $\exists\, x^* \in (a, b)$ s.t. $f'(x^*) = \dfrac{f(b) - f(a)}{b - a}$

$$h(x) = \frac{(f(b) - f(a))\, x - (b - a)\, f(x)}{t}$$