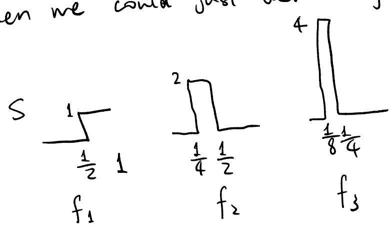
What if every f were a <u>limit</u> of $f_1, f_2, f_3, \ldots$ step functions

Then we could just define $\int f = \lim_{i \to \infty} \int f_i$



$f_i \to 0$ pointwise

$\int_0^1 f_i(x)\,dx = \frac{1}{2} \quad \forall i$

Let f be a bounded function on $[a, b]$

A partition of $[a,b]$ is a "finite set of real #s"

$$a = x_0 \le x_1 \le \cdots \le x_n$$

$$\Delta x_i = x_{i+1} - x_i$$

Two ways to approximate f by a step function

$$M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$$

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

exist because f is bounded

$$U(P, f) = \sum M_i \Delta x_i$$

$$L(P, f) = \sum m_i \Delta x_i$$

Def Upper intergral of f on $[a, b]$ is $\inf\limits_P U(P,f)$

lower intergral of f on $[a, b]$ is $\sup\limits_P L(P, f)$

Prop lower integral $\le$ upper integral

Pf $\sup\limits_P L(P,f) \le \inf\limits_P U(P, f)$

$\Leftrightarrow$

$L(P, f) \le U(P, f)$

for every pair of pointwise P_1, P_2

recall: S, T sets of reals

$\sup S \le \inf T$

$\Leftrightarrow$

every element of S $\le$ every element of T

Def We say P^* is a refinement of P if every breakpoint of P is a breakpoint of P^*

P[| | |] We say $P^* < P$

P*[| | | | | | | | |]

Lemma If P^* is a partition of P, then $U(P^*, f) \le U(P, f)$, $L(P^*, f) \ge L(P, f)$

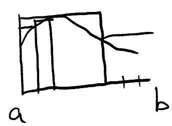
?

Fact If P_1 and P_2 are partitions, there is a partition Q which is a common refinement of both

$Q < P_1$

$Q < P_2$

(Simply take the union of breakpoints$(P_1) \cup$ breakpoints(P_2))



$$L(\ , \ ) \le L(\ , \ ) \le U(Q, f) \le U(?, \ )$$

Def When lower integral = upper integral, we say f is (Riemann) <u>integrable</u> on $[a, b)$ and define $\int_a^b f(x)\,dx$ to be the common value

Analogy:

upper int ← lim sup ← always exist

lower int ← lim inf ←

$\int$ lim

A non-integrable function

$\chi_{\mathbb{Q}} : [a, b] \to \mathbb{R}$

$$\chi(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

$U(P, \chi_{\mathbb{Q}}) = b - a =$ upper integral

$L(P, \chi_{\mathbb{Q}}) = 0 =$ lower integral

a b

On the other hand, consider

$\chi_{\text{Cantor set}} : [0, 1] \to \mathbb{R}$

P_1 $0 \quad \frac{1}{3}+.001 \quad \frac{2}{3}-.001 \quad 1$

P_2

P_k

$U(P_1, f) = \frac{2}{3} + 2 \cdot (.001)$

$U(P_2, f) = \frac{4}{9} + 2 \cdot (.001)^2 + 4 \cdot (.001)^2$

$U(P_k, f) = \left(\frac{2}{3}\right)^k + \text{small}$

$\inf_P U(P, f) \leq \inf_k U(P_k, f) = 0$

so $\inf_P U(P, f) = 0$

$\int_0^1 \chi_{\text{Cantor set}}(x)\,dx = 0$

P

inf $\chi_{\text{Cantor set}}$ on this interval is always 0

Thm If f is continuous, then f is integrable

Thm If f is monotone, then f is integrable

Facts If $c \in [a, b]$ and f integrable on $[a, b)$ then

$$\int_a^b f(x)\,dx - \int_a^c f(x)\,dx = \int_c^b f(x)\,dx$$

If f is integrable, $|f|$ is integrable, and

$$\int_a^b |f(x)|\,dx \geq \left|\int_a^b f(x)\,dx\right| \quad \text{“}\Delta\text{ inequality”}$$

INDEFINITE INTEGRALS

Let f integrable on $[a,b]$. Then we can define $F:[a,b]\to\mathbb{R}$ by

$$F(x)=\int_a^x f(t)\,dt$$

differentiations makes functions worse ?

Thm F is continuous

Pf I need to show that $y\to x \Rightarrow F(y)\to F(x)$

$$|F(y)-F(x)| = \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right|$$

$$\le \left|\int_x^y f(t)\,dt\right|$$

$$\le \int_x^y \Big(\underbrace{\sup_t |f(t)|}_{S:=}\Big)\,dt = S\,|y-x|$$

(In fact, F is Lipschitz)

FTC ?

If f is continuous at $x_0\in[a,b]$ and integrable then F is differentiable at x_0, and $F'(x_0)=f(x_0)$

We need to show $\displaystyle\lim_{y\to x_0}\Big(\underbrace{\frac{F(y)-F(x_0)}{y-x_0}-f(x_0)}_{(*)}\Big)=0$

i.e., given ε, I can choose δ s.t. $|y-x|<\delta \Rightarrow (*)<\varepsilon$

$$F(y)-F(x_0)=\int_{x_0}^y f(t)\,dt$$

$$\left|\frac{1}{y-x_0}\int_{x_0}^y f(t)\,dt - \frac{1}{y-x_0}\int_{x_0}^y f(x_0)\,dt\right| = \left|\frac{1}{y-x_0}\int_{x_0}^y \big(f(t)-f(x_0)\big)\,dt\right|$$

Because f is continuous at x_0, we may choose δ s.t.

$$|y-x_0|<\delta \Rightarrow |f(y)-f(x_0)|<\varepsilon$$

Do so

So y lies in $(x-\delta,\ x+\delta)$

$($ $x_0-\delta$ $\cdot$ x_0 $\cdot$ t $\cdot$ y $x_0+\delta$ $)$

so $|f(t)-f(x_0)|<\varepsilon$ for all $t\in[x_0, y]$

by triangle inequality,

$$\le \frac{1}{y-x_0}\int_{x_0}^y |f(t)-f(x_0)|\,dt$$

$$\le \frac{1}{y-x_0}\int_{x_0}^y \varepsilon\,dt < \varepsilon \quad \checkmark$$