COMPLETE ORDERED FIELD

FIELDS & ORDERED FIELDS

Roughly speaking a field is a set where we can do arithmetic ($+, -, \times, \div$)
$\div$ except 0

e.g. $\mathbb{Q}$

e.g. Boolean field, $\mathbb{F}_2$, $\mathbb{Z}/2\mathbb{Z}$

$\{0, 1\}$ "modular arithmetic mod 2"

$0+1=1$ $\quad$ $1+1=0$

$0-1=1$

$0\cdot 1=1$

$1\cdot 1=1$

Non-e.g.: $\mathbb{Z}$ because you cannot compute $\frac{1}{2}$

To be slightly more precise, we require the existence of 0, 1 and that the operations obey "reasonable" rules

$$0+x=x$$
$$x+y=y+x$$
$$(x+y)+z=x+(y+z)$$
$$x(y+z)=xy+xz$$
$$xy=yx$$

ORDERED FIELD

If S is an ordered set, and S is a field, it is not necessarily the case that S is an ordered field

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Def An ordered field is a field F with an ordering $<$ such that

- For $x, y, z \in F$ with $y<z$, $\quad x+y<x+z$
- If $x, y \in F$ $x>0$, $y>0$, $\quad xy>0$

e.g. $\mathbb{Q}$ is an ordered field

Non-e.g. there is no ordering on $\mathbb{F}_2$ that makes it an ordered field

$\{0, 1\}$ Suppose $1 < 0$

then $1+1 < 1+0$

$0 < 1$

FACT: In an ordered field, $x^2 \geq 0 \quad \forall x \in F$

Case 1: $x > 0$, $x \cdot x > 0$, $x^2 > 0$ done

Case 2: $x = 0$, $x^2 = 0$ done

Case 3: $x < 0$ so $[(-1)x]^2 > 0$

$x - x < 0 - x$ $\quad (-1)^2 x^2 > 0$

$0 < (-1)x$ $\quad x^2 > 0$

Last time we introduced the notion: $E \subset S$

$x \in S$ is a least upper bound if

- s is an upper bound for E
- If t is an upper bound for E, $t \geq s$

In this case, we call s $\sup E$, or the supremum of E (in S)

FACT A bounded ordered set $E \subset S$ has at most one supremum

PF Suppose s, s' are suprema of E

We need to prove $s = s'$

s' is a supremum $\Rightarrow$ s' is an upper bond for E

Since s is a supremum, $s' \geq s$

Similarly, (mutatis mutandis) $s \geq s'$ $\Rightarrow$ $s = s'$ ✓

Similarly, greatest lower bound for E is called <u>infimum</u> $\inf E$

e.g. $E \subset \mathbb{Q} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

$\left\{\frac{1}{n}\right\}_{n \in \mathbb{Z}_{>0}}$

Does E have a sup? an inf? What are they?

$1 \qquad 0$

Unfortunately, not every bounded E has a supremum

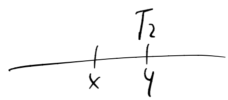
<u>Example</u> Let $A \subset \mathbb{Q}_{>0}$ be the set $\{x \in \mathbb{Q}_{>0} : x^2 < 2\}$

<u>FACT</u> A has no supremum

<u>Pf</u> The first step is to describe what the upper bounds for A are

<u>Lemma</u> The upper bounds for A are these numbers $y \in \mathbb{Q}$ s.t. $y^2 > 2$

Warmup: If $y^2 > 2$, then y is an upper bound for A



<u>Pf</u>: If $x \in A$,

we need to show $y > x$

what we know is that $y^2 > x^2$

So $y^2 - x^2 > 0$

$(y-x)(y+x) > 0 \Rightarrow y - x > 0$ (divide both sides by $y+x$) which is positive

What we need to show is that there are no <u>other</u> upper bounds for A

i.e. we need to show that if $y^2 < 2$

then y is <u>not</u> an upper bound for A

say $y = 1.4$, $y^2 = 1.96$

In other words, we want to find x s.t.

$x \in A$ (i.e. $x^2 < 2$) and $x > y$

positive

Let's consider $x^2 - y^2 = (x+y)(x-y)$

(which we know is positive)

we want to find x s.t. $x - y > 0$

and $x^2 < 2$, i.e. $x^2 - y^2 < 2 - y^2$

Intuition we just want to make $x-y$ really really small

We may assume that $x, y < 100$

Thus $x^2 - y^2 < (x+y)(x-y) < 200(x-y)$

I want: $x^2 - y^2 < 2 - y^2$

I win if: $200(x-y) < 2 - y^2$

equivalently $x - y < \frac{2-y^2}{200}$ $\quad x < \frac{2-y^2}{200} + y$

$z = x - y$ $\quad 0 < z < \frac{2-y^2}{200}$

$$z = \frac{1}{2}\left(\frac{2-y^2}{200}\right)$$

$$x = \left(\frac{2-y^2}{200}\right) + y$$