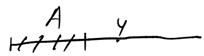
MATH 521 Lecture 4 20 Sep 2022 R. T.

Last time:

$A = \{x \in \mathbb{Q}_{>0} : x^2 < 2\}$



We were proving that A has no supremum

We had just proven: y is an upper bound for A iff $y^2 \geq 2$

Now, let $B = \{y \in \mathbb{Q}_{>0} : y^2 > 2\}$

Claim: if $y \in B$, then y is not an least upper bound for A
(though it is an upper bound for A)

To prove this, need to show that, given $y \in B$,
there is $x < y$ $x^2 > 2$ (so x is also an UB for A)

This is mutatis mutandis the same last time

So if y is a least upper bound for A, then $y^2 \geq 2$ (Thursday)
but not $y^2 > 2$ (just now)
$\Rightarrow y^2 = 2$

<u>BUT</u>: there is no solution in $\mathbb{Q}$ to $y^2 = 2$

<u>Pf</u>: Suppose a, b are integers with

$$\left(\frac{a}{b}\right)^2 = 2$$

$$a^2 = 2b^2$$

a^2 is even, so a is even, $a = 2c$

$$(2c)^2 = 2b^2$$
$$4c^2 = 2b^2$$
$$2c^2 = b^2$$

b^2 is even, so b is even

So write $b = 2d$

$$2c^2 = (2d)^2$$
$$2c^2 = 4d^2$$
$$c^2 = 2d^2$$

Conclusion: A has no least upper bound

Def We say a nonempty set S has the <u>least upper bound</u> <u>property</u> if every nonempty $E \subset S$ which is bounded above has a least upper bound

Why do I specify nonempty?

Q: Let $S = \{1, 2, 3, 4, 5\}$ $E = \emptyset$

Does $\emptyset$ have a least upper bound? greatest lower bound?

is it bounded above (below)?

An upper bound is s s.t. $s \geq x$ for all $x \in \emptyset$

s <u>not</u> an upper bound iff $\exists x \in \emptyset$ $s < x$

$\sup \emptyset = 1$ $\inf \emptyset = 5$

<u>Thm</u> Suppose S has LUBP. Then S has GLBP

I need to show it has a greatest lower bound

Let A be the set $\{x \in S : x \text{ is a lower bound for } B\}$ $\forall a \in A, b \in B$ $a \leq b$

A is nonempty (because B bounded below)

A is bounded above (because B is nonempty)

So by LUBP, A has a supremum $\alpha = \sup A$

Claim: α is the infimum for B

First of all, need to check α is a lower bound

Suppose not. Then $\exists b \in B$ with $b < \alpha$

But remember every $b \in B$ is an upper bound for A $\times$ because $\alpha = \sup A$

Now to prove α is the <u>greatest</u> lower bound for B

Suppose not:

Let β be another lower bound for B

So $\beta \in A$, so $\alpha \geq \beta$ because α is an UB for A

So $\alpha \geq$ any upper bound for $B \Rightarrow \alpha = \inf B$

Def We say an ordered field is complete if it has the least upper bound property

Rk $\mathbb{Q}$ is not complete

A finite ordered set has the LUBP

$\{1, 2, 3, 4, 5\}$

Thm There is a complete ordered field
(We are going to call it $\mathbb{R}$)

Why is the product of no numbers 1?

product([]) $\prod_{x \in \phi} x$

S, T disjoint sets of #s

"product of all numbers in $S \cup T$"

= (product of S) . (product of T)

$\{1, 2\}$ 2

$\{4, 5\}$ 20

$\{1, 2, 4, 5\}$ 40

{product of $S \cup \emptyset$} = {product of S}

= {product of S} . {product of ϕ}

$\frac{3!}{3} = 2!$ $\frac{2!}{2} = 1!$ $\frac{1!}{1} = 0$

SLIGHT ALG DIGRESSION

Recall: A bijection between sets S, T is
$f: S \to T$ which is injective and surjective

If F, G are ordered fields, we can define an isomorphism to be a function such that

- f is a bijection
- $f(x) > f(y)$ iff $x > y$
- $f(x+y) = f(x) + f(y)$
- $f(xy) = f(x) f(y)$

We can think of F and G being "the same field with the elements labelled differently"

What we will prove:

$\exists$ a complete ordered field $\mathbb{R}$

won't prove:

If F is any complete ordered field,

$\exists$ $\boxed{?}$ $f: F \to \mathbb{R}$