MATH 521 Lecture 5 22 Sep 2022 R.T.

COMPLETE ORDERED FIELD(S)

Prop Let F be a COF. The F has the Archimedean property.
for any $x, y > 0$ $\exists n \in \mathbb{Z}_{>0}$ s.t. $nx > y$

Pf Suppose otherwise i.e. $nx \leq y$ for all $n \in \mathbb{Z}_{>0}$

$E: x, 2x, 3x, 4x, \ldots \leq y$

By LUBP, E has a supremum $\lambda = \sup E$

We derive a contradiction: $\lambda - x < \lambda$

so $\lambda - x$ is not an upper bound for E

So there is some element of E which is $> \lambda - x$

i.e. for some m $mx > a - x$

So $(m+1)x > \lambda$ X because λ is an UB for E

Prop Let F a COF, n a positive integer, $x > 0$ in F

Then there is a unique $y \in F$ with $y^n = x$ "n^{th} roots exist"

Pf Let $T_{n,x} = \{z : z \geq 0, z^n \leq x\}$ $\sqrt[n]{x}$ ~~[illegible]~~

$T_{n,x}$ is nonempty ($0 \in T_{n,x}$)

$T_{n,x}$ is bounded above (by ?)

Is it the case that $x \geq z$ for all $z \in T_{n,x}$?

Not necessarily e.g. $T_{2,\frac{1}{4}} = \{z : z^2 \leq \frac{1}{4}\}$

This includes $t = \frac{1}{3}$ So $\frac{1}{4}$ is not an upper bound for $T_{2,\frac{1}{4}}$

Lemma a is an upper bound for $T_{n,x}$ iff $a^n \geq x$

If $x \geq 1$, claim $x^n \geq x$

Pf $x^n - x = x(x^{n-1} - 1) = \underset{\text{positive}}{x}\ \underset{\text{non-negative}}{(x-1)}\ (1 + x + \underset{\text{positive}}{x^2} + \cdots + x^{n-2})$

If $x < 1$, $T_{n,x}$ is bounded above by 1

So let $\lambda = \sup T_{n,x}$

We will show $\lambda^n = x$

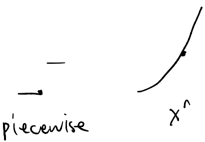
Two steps: $\lambda^n \leq x$ ← we will just do this

$\lambda^n \geq x$

Suppose, on the contrary, $\lambda^n < x$

To derive a contradiction, we want to find some λ' such that

$\lambda' > \lambda \quad \lambda' \in T_{n,x}$



i.e. choose $\varepsilon > 0$ such that $(\lambda + \varepsilon) \in T_{n,x}$

i.e. $(\lambda+\varepsilon)^n \leq x$

$$(\lambda+\varepsilon)^n = \lambda^n + n\lambda^{n-1}\varepsilon + \binom{n}{2}\lambda^{n-2}\varepsilon^2 + \cdots + \varepsilon^n$$

We want:

$$\lambda^n + \cdots + \varepsilon^n < x$$

$$n\lambda^{n-1}\varepsilon + \binom{n}{2}\lambda^{n-2}\varepsilon^2 + \cdots + \varepsilon^n < x - \lambda^n$$

↑ note this is positive

$$\varepsilon\left(n\lambda^{n-1} + \binom{n}{2}\lambda^{n-2}\varepsilon + \cdots + \varepsilon^{n-1}\right) < x - \lambda^n$$

We need to choose ε s.t. $\varepsilon < \dfrac{x-\lambda^n}{(n\lambda^{n-1} + \cdots + \varepsilon^n)}$

Let us suppose $\varepsilon < 1$

Then $n\lambda^{n-1} + \binom{n}{2}\lambda^{n-2}\varepsilon + \cdots + \varepsilon^n \leq \underbrace{n\lambda^{n-1} + \binom{n}{2}\lambda^{n-2} + \cdots + 1}_{= C_{n,\lambda}}$

Now $\dfrac{x-\lambda^n}{C_{n,\lambda}} \leq \dfrac{x-\lambda^n}{(n\lambda^{n-1} + \cdots + \varepsilon^n)}$

So choose $\varepsilon < \dfrac{x-\lambda^n}{C_{n,\lambda}}$

To sum, we showed that if $\lambda = \sup T_{n,x}$ then $\lambda^n \geq x$

$\mathcal{M}\mathcal{M}$, $\lambda^n \leq x \Rightarrow \lambda^n = x$

Uniqueness: Suppose y, y' satisfy $y^n = (y')^n = x$

To prove: $y = y'$

If $(y')^n = y^n$, $(y')^n - y^n = 0 \Rightarrow$

$$(y'-y)\left[(y')^{n-1} + \underline{(y')^{n-2}y} + \cdots + y'y^{n-2} + y^{n-1}\right] = 0$$

In any field, $xy = 0 \Rightarrow$ either $x = 0$ or $y = 0$

because if $x \neq 0, y \neq 0$, $\underset{\underset{0}{\shortparallel}}{xy} \cdot \frac{1}{x} \cdot \frac{1}{y} = 1$

$(y')^{n-2}y$ is positive, since y and y' are

So $y' - y = 0$

CONSTRUCTION OF $\mathbb{R}$

1, 1.414, 1.4142, ...

A real number is a sequence of rational numbers

1, 2, 4, 8, 16, ...

A bounded sequence? 1, -1, 1, -1, 1, -1

We want to say a sequence is convergent without talking about what it converges to

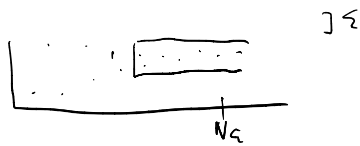
Insight of Cauchy: if the terms of a sequence are getting closer and closer to ????, they are getting closer and closer to each other

Def A Cauchy sequence in $\mathbb{Q}$ is a sequence $a_1, a_2, a_3, \ldots$ such that

$\forall \varepsilon > 0$, there exists $N \in \mathbb{Z}_{>0}$ such that,

for all $m, n > N$, $|a_m - a_n| < \varepsilon$

Picture



Cauchy Sequence: 1, 1.4, 1.41, 1.414, ...

Not Cauchy Sequence: 1, -1, 1, -1, ... $\varepsilon = \frac{1}{2}$

A real number is a Cauchy sequence

$\sqrt{2} = 1, 1.4, 1.41, \ldots$

$2 = 2, 2, 2, 2, 2, 2, \ldots$

$2' = 3, 2.1, 2.01, 2.001, 2.0001, \ldots$

$2' - 2 = 1, 0.1, 0.01, 0.001, \ldots$

$$\frac{1}{2^{i-2}} = 1, 10, 100, 1000, \dots$$

Not a Cauchy Sequence