MATH 521 Lecture 6

Last time

Def A <u>Cauchy sequence</u> of rationals is

$a = a_1, a_2, \ldots$ s.t. for all $\varepsilon > 0$, $\exists N$, $\forall i, j > N$

$|a_i - a_j| < \varepsilon$

Proposal a real number is a Cauchy sequence

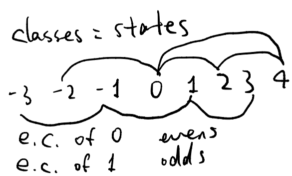
e.g. $2, 2, 2, 2, 2, \ldots$ and $0, 0, 2, 2, 2, \ldots$

We really want these to be the same real number (2)

but they are <u>not</u> the same sequence

Def An <u>equivalence relation</u> on S is a relation $\sim \in S \times S$ such that

- $\forall x \quad x \sim x$
- $\forall x, y \quad x \sim y \Leftrightarrow y \sim x$
- $\forall x, y, z \quad x \sim y$ and $y \sim z \Rightarrow x \sim z$



e.g. S cities $\quad \sim$ "in the same state" ✓

S $\mathbb{Z}$ $\quad \sim$ $m \sim n$ iff $m - n$ is even ✓

$m - p = (m-n) + (n-p)$

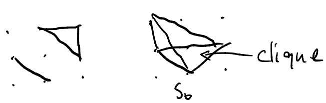
S $\mathbb{Z}$ $\quad \sim$ $m \sim n$ iff $m - n$ is odd ✗

S Sets $\quad \sim$ $S \sim T$ if $\exists$ a bijection $f: S \to T$ ✓

S $\mathbb{Q}$ $\quad \sim$ $x \sim y$ if $|x - y| < 0.01$ ✗

$x = 0.001 \quad y = 0.009 \quad z = 0.011$

EQUIVALANCE CLASSES



An equation relation $\sim$ partitions S into disjoint subsets S_i, each one of the form $\{s \in S : s \sim s_0\}$ for some s_0

Def Two Cauchy sequences a, b are equivalent if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall i$ $|a_i - b_i| < \varepsilon$

2, 2, 2, 2, ...
0, 0, 2, 2, ...

3, 2.1, 2.001, ...
2, 2, 2, ...

Def A real number is an equivalence class of Cauchy sequences under this relation

Comment: this is one way to define $\mathbb{R}$ cuts (Rudin, Leslie)
sequence (rules of Kemp)
There is a lot to check e.g. why is Cauchy equivalence and equivalence relation?

Prop If $\underline{a}, \underline{b}, \underline{c}$ are Cauchy sequences, $\underline{a} \sim \underline{b}$ and $\underline{b} \sim \underline{c}$, then $\underline{a} \sim \underline{c}$

Pf
$a_1, a_2, a_3, a_4, a_5, \ldots$
$b_1, b_2, b_3, b_4, b_5, \ldots$
$c_1, c_2, c_3, c_4, c_5, \ldots$

Pf $\exists N_1$ s.t. $|a_i - b_i| < \frac{\varepsilon}{2}$ for all $i > N_1$
$\exists N_2$ s.t. $|b_i - c_i| < \frac{\varepsilon}{2}$ for all $i > N_2$
So for all $i > \max(N_1, N_2)$
$$|a_i - c_i| < \varepsilon$$

Pf $\exists N_{a,b}, \varepsilon_{a,b}$ s.t. $|a_i - b_i| < \varepsilon_{a,b}$ for all $i > N_1$
$\exists N_{b,c}, \varepsilon_{b,c}$ s.t. $|b_i - c_i| < \varepsilon_{b,c}$ for all $i > N_2$
So for all $i > \max(N_1, N_2)$
$$|a_i - c_i| < \varepsilon_{a,b} + \varepsilon_{b,c}$$

Choose $\varepsilon_{a,b}$ and $\varepsilon_{b,c}$ s.t.
$$\varepsilon_{a,b} + \varepsilon_{b,c} < \varepsilon_{a,c} \quad \checkmark$$

$$\sqrt{2} = \begin{cases} 1, 1.4, 1.414, \ldots \\ 5, \frac{16}{17}, -18, 1, 1.4, 1.414, 1.4145, \ldots \\ \vdots \end{cases}$$

Thm $\mathbb{R}$ is a complete ordered field
(see Kemp's notes for full proof)

FIELD

We have to show we can do arithmetic, e.g. what is $\times$?

$$\underline{a}\,\underline{b} = (a_1 b_1,\ a_2 b_2,\ \dots)$$

Needs to be proved that if $\underline{a}$ and $\underline{b}$ are Cauchy sequences, then $\underline{ab}$ is also Cauchy – i.e. $\forall \varepsilon > 0,\ \exists N,\ \forall i, j > N$

$$|a_i b_i - a_j b_j| < \varepsilon$$

↑ small

How do you know you can write

$(0.1000000 - (.001) \cdot 1000000.001)$

and if $\underline{a} \sim \underline{a}'$, $\underline{b} \sim \underline{b}'$ then $\underline{a}\,\underline{b} \sim \underline{a}'\underline{b}'$

ORDERED

What should it mean to say $\underline{a} > \underline{b}$?

Bad: (1) $a_i > b_i\ \forall i$

(2) $\exists N$ s.t. $a_i > b_i\ \forall i > N$

But consider

$a = 3, 2, 1, 2.01, 2.001, \dots$

$b = 2, 2, 2, 2, \dots$

Def We say $\underline{a} > \underline{b}$ if $\exists \delta > 0, N$ such that $a_i > b_i + \delta$ for all $i > N$

$\underline{a} = 0$

$\underline{a} > 0$

"bounded away" from 0

Prop If x, y are real numbers,
exactly one of $x<y$, $x>y$, or $x=y$ is true

One can now define

Def A Cauchy sequence of real numbers is a sequence
$x_1, x_2, x_3, \dots$ $\quad x_i \in \mathbb{R}$
s.t. for any $\varepsilon > 0$, $\exists N$ s.t. $\forall i,j > N$ $|x_i - x_j| < \varepsilon$

Def A sequence of reals $x_1, x_2, \dots$ has <u>limit y</u> if, $\forall \varepsilon > 0$,
$\exists N$ s.t. $\forall i > N$ $|x_i - y| < \varepsilon$

Thm $\mathbb{R}$ has the least upper bound property

pf Let $S \subset \mathbb{R}$ be a nonempty subset, bounded above
Let $s \in S$ (because S nonempty)
Let $M \in \mathbb{R}$ an upper bound for S, $M \geq s$

Iterative process starting from $(l_0, u_0) = (s, M)$

- Let m be average of l_i, u_i
- If m is an upper bound for S, replace u_i with m
- If m is not an UB for S, replace l_i with m

Repeat

This yields $(s, M) = (l_0, u_0), (l_1, u_1), (l_2, u_2), \dots$

$u_i - u_{i+1}$ $\qquad$ $u_i \searrow u_{i+1}$

$l_i \nearrow l_{i+1}$ $\qquad$ $l_i - l_{i+1}$

CLAIM: Both $\underline{\ell}$ and $\underline{u}$ have limits,
the limits are the same, and this limit is $\sup S$