MATH 521 Lecture 7

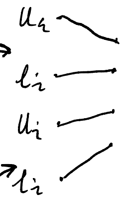
We are proving $\mathbb{R}$ has the LUBP

$S \subset \mathbb{R}$ nonempty, bounded above $< M$

$\forall s \in S \quad s \leq M$

$(l_0, u_0) = (s, M)$

- m = average of l_i, u_i
- If m is an UB for S, $(l_{i+1}, u_{i+1}) = (l_i, m)$
- Otherwise, $(l_{i+1}, u_{i+1}) = (m, u_i)$



<u>Properties</u> $u_i > l_i \ \forall i$

$\underline{u}$ is non-increasing

$\underline{l}$ is non-decreasing

$\underline{u}$ bounded above by M

$\underline{l}$ bounded below by s

$\forall i \ u_i$ is an upper bound for S

$\forall i \ l_i$ is <u>not</u> an upper bound for S

(unless s is an UB for S, but then $s = \sup S$ done)

We will prove:

- $\underline{u}$ and $\underline{l}$ both have limits, the limits are same, and they are $\sup S$

<u>Prop</u> ($\mathbb{R}$ is complete) Every Cauchy sequence $\underline{x} = x_1, x_2, x_3, \ldots$ has a limit

indeed, a sequence in $\mathbb{R}$ has a limit <u>iff</u> it is Cauchy

Main idea: $\forall i$ choose $a_i \in \mathbb{Q}$ s.t. $|a_i - x_i| \leq \frac{1}{2^i}$

$\underline{a} = a_1, a_2, a_3, \ldots$

Then prove $\underline{a}$ is Cauchy

Take y to be the real number represented by $\underline{a}$

Then prove $\lim \underline{x} = y$

for this last step: $\underline{a} - \underline{x}$ is a sequence with $|a_i - x_i| < \frac{1}{2} i$

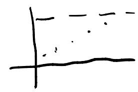
So $\lim(a - x) = 0$

$$\underbrace{\lim(a-x)}_{\lim a - \lim x} = 0$$

To show $\underline{\ell}$ has a limit, we will show

Prop (Monotone Convergence Thm)

A non-decreasing sequence of real #s which is bounded above is Cauchy (and thus converges to a limit)



EVERYTHING THAT RISES MUST CONVERGE

Pf $\underline{x}$ an non-decreasing sequence, bounded by some M

Suppose $\underline{x}$ is not Cauchy

$\underline{x}$ Cauchy would mean

$$\forall \varepsilon > 0, \ \exists N : \underbrace{\forall i,j > N, \ |x_j - x_i| < \varepsilon}_{N \text{ is } \varepsilon\text{-green}}$$

" $\varepsilon > 0$, there exists an ε-green N "

Suppose $\exists \varepsilon > 0$ for which there is no ε-green N

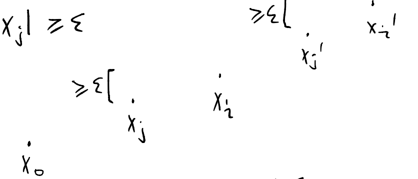
$\Leftrightarrow \exists \varepsilon > 0$ such that $\forall N$, N is not ε-green

$\exists \varepsilon > 0, \ \forall N, \ \exists i,j > N \quad |x_i - x_j| \geq \varepsilon$

$i \geq j$

WLOG

without loss of generality



$N = 0 \quad x_i \geq x_j + \varepsilon \geq x_0 + \varepsilon$

$N = x_i \quad x_{i'} \geq x_{j'} + \varepsilon \geq x_i + \varepsilon \geq x_0 + 2\varepsilon$

$N = x_{i'} \ \ldots$

There by this process (induction) for every $m \in \mathbb{Z}_{>0}$

$\exists I$ s.t. $x_I \geq x_0 + m\varepsilon$

But $x_i < M$ for all i ↙ Archemedean property

Choose m s.t. $m\varepsilon > M - x_0$

$x_0 + m\varepsilon > M$ X

Back to Proof of LUBP:

By Monotonic Convergence,

$\underline{\ell}$ has a limit mutchs mutadis

$\underline{u}$ has a limit

$\ell_i - u_i = \frac{1}{2^i}(M - s)$

So $\lim(\underline{\ell} - \underline{u}) = 0$, so $\lim \underline{\ell} = \lim \underline{u}$

Let $u = \lim \underline{u} = \lim_{i\to\infty} u_i$

To show: $u = \sup S$

First: show u is an UB for S

Recall: each u_i is an UB for S

Suppose u is <u>not</u> an UB for S, i.e., there is $s \in S$ s.t. $u < s$

s ·
u_i ·
u ·

Let $\varepsilon = \left(\frac{1}{2}\right)(s-u)$

Choose i s.t. $|u_i - u| < \varepsilon$

then $u_i < s$ X

To show u is the <u>least</u> upper bound,

Suppose $v < u$ is also an UB for S

Recall $u = \lim \underline{\ell}$

u ·
ℓ_i ·
v ·

Choose ℓ_i s.t. $|\ell_i - u| \leq \frac{1}{2}(u-v)$

So $\ell_i > v$

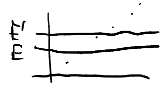
So ℓ_i is an UB for S X

Extension of $\mathbb{R}$

$\overline{\mathbb{R}}$, the "extended reals", an ordered set $\mathbb{R} \cup \{\infty, -\infty\}$

Def If $x_1, x_2, \ldots$ is a sequence in $\overline{\mathbb{R}}$, we say $\lim \underline{x} = \infty$ if

$\forall E, \exists N, \forall i > N, x_i > E$



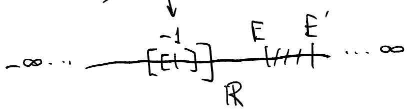
cross bound and never back

choosing ε is choosing smaller and smaller neighborhoods of -1

$[E, \infty]$ is a neighborhood of ∞

$\cup$

$[E', \infty)$ is a smaller neighborhood of ∞



Problem with $\overline{\mathbb{R}}$ – not a field

$\infty = \lim 1, 2, 3, 4, 5, \ldots$

$1 = \lim 1, 1, 1, 1, \ldots$

$\infty + 1 = \lim 2, 3, 4, 5, \ldots = \infty$

$\infty + 1 = \infty$ ☹

$1 = 0$

INFINITESIMALS

$\varepsilon = \sqrt{0}$

Including this in the reals, we get numbers of the form

$a + b\varepsilon \qquad a, b \in \mathbb{R}$

$(a+b\varepsilon)(c+d\varepsilon) = ac + (b+d)\varepsilon$

$\varepsilon < x$ for any $x > 0$

$\varepsilon > 0$

$a + b\varepsilon > c + d\varepsilon$ if either $a \geq c$ or $(a = c$ and $b \geq d)$

Not a field because ε has no reciprocal

pf $\quad \varepsilon(a + b\varepsilon) = 1$

$0 + a\varepsilon = 1 + 0\varepsilon$