MATH 521 Lecture 8

CARDINALITY:

Recall: we defined an equivalence relation on sets

$S \sim T$ if $\exists f : S \xrightarrow{\sim} T$
which is a bijection

Note if $f : S \xrightarrow{\sim} T$, then $f^{-1} : T \xrightarrow{\sim} S$

Transitivity
$$\begin{tikzcd} S \arrow[r, "\sim", "f"'] \arrow[rr, bend left, "gf"] & T \arrow[r, "\sim", "g"'] & U \end{tikzcd}$$
check gf is a bijection

The equivalence classes under $\sim$ are called <u>cardinalities</u>

<u>FACT</u>: If S and T are finite sets, $S \sim T$ iff $|S| = |T|$

"pidgeonhole principle"

This means that the equivalence classes of finite sets are <u>natural numbers</u>

e.g. $\{\{1,2,3,4\}$, compress directrum, Beatles, wheels on my car, $\ldots\} = 4$

Also for finite sets,

$|S| < |T|$ iff there is an injection $f : S \hookrightarrow T$ which is not a bijection

($|S| > |T|$: no injection at all
$|S| = |T|$: every injection is a bijection)

BUT: $\mathbb{Z} \to \mathbb{Z}$ is an injection
$n \mapsto 2n$ but not a bijection

<u>Def</u> A set S is <u>infinite</u> if $\exists : f : S \hookrightarrow S$ an injection which is not a bijection

Fact An infinite set cannot be equivalent to a finite set

pf: Suppose S is infinite

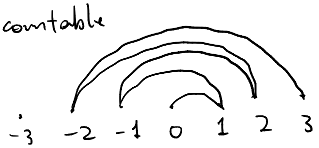
$$\begin{tikzcd} S \arrow[r, leftrightarrow, "f"] & S \arrow[d, "\phi^{-1}"] \\ F \arrow[u, "\phi"] \arrow[r, bend right, "\phi^{-1}f\phi"'] & F \end{tikzcd}$$

Suppose F is a set with $F \sim S$

You could prove $\phi^{-1} f \phi$ is injective but not bijective $\Rightarrow$ F is infinite

Def We say S is <u>countably infinite</u> if $S \sim \mathbb{Z}_{>0}$

Prop $\mathbb{Z}$ is countable



This function depicts a bijection $f: \mathbb{Z}_{\geq 0} \to \mathbb{Z}$

f injective: flea never hits the same number twice

f surjective: every number is eventually hit by the flea

$f(0) = 0$

$f(1) = 1$

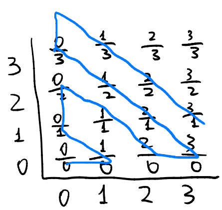
$f(2) = -1$

$f(3) = 2$

$f(4) = -2$

$\vdots$

Prop $\mathbb{Q}_{\geq 0}$ is countable



$f(0) = \frac{0}{1}$

$f(1) = \frac{1}{1}$

$f(2) = \frac{2}{1}$

$f(3) = \frac{1}{2}$

$f(4) = \frac{1}{3}$

$f(5) = \frac{3}{1}$

$\vdots$

Useful facts:

If S, T are countable, so is $S \cup T$

$s_1, s_2, s_3, \ldots$

$t_1, t_2, t_3, \ldots$

If S countable, any subset of S is countably infinite or finite

$s_1, s_2, s_3, s_4, \ldots$

If S, T are countable, so is $S \times T$ e.g. $\mathbb{Z}^{100}$ is countable

Cantor (1880s) there are uncountable sets

Notation: if S and T are sets, we denote by S^T the set of functions from T to S

e.g. $\{0,1\}^{\text{Beatles}}$ = set of all functions from Beatles to $\{0,1\}$

Paul	1
Ringo	0
John	0
George	1

is an element of $\{0,1\}^{\text{Beatles}}$

$|\{0,1\}|^{\text{Beatles}} = 16 = 2^4 = |\{0,1\}|^{|\text{Beatles}|}$

In general, if S, T finite $\quad |S^T| = |S|^{|T|}$

Note also: $\{0,1\}^T$ = set of subsets of T

What is $\{0,1\}^{\emptyset}$? $\{$functions from $\emptyset$ to $\{0,1\}\}$

A function from $\emptyset$ to $\{0,1\}$ is a subset of $\emptyset \times \{0,1\}$ satisfying [rules]

$$\emptyset \times \{0,1\} = \left\{(a,b) \;\; \begin{matrix} a \in \emptyset \\ b \in \{0,1\}\end{matrix}\right\} \qquad \emptyset \times \{0,1\} = \emptyset$$

A function $f: \emptyset \to \{0,1\}$ is a subset of $\emptyset$ satisfying [rules]

$\emptyset$ is the only subset of $\emptyset \qquad |\{0,1\}^{\emptyset}| = 1 = 2^0 = |\{0,1\}|^{|\emptyset|}$

<u>Thm</u> (Cantor) $\{0,1\}^{\mathbb{Z}_{\geq 0}}$ is uncountable

$\{0,1\}^{\mathbb{Z}_{\geq 0}}$ = set of subsets of $\mathbb{Z}_{\geq 0}$ = "power set" of $\mathbb{Z}_{\geq 0}$

Pf (Cantor's diagonal argument)

Suppose $f: \mathbb{Z}_{\geq 0} \xrightarrow{\sim} \{0, 1\}$

$D: 1\ 1\ 1\ 0 \cdots$

$f(0): \boxed{0}111\ 01000 \cdots$

$f(1): 1\boxed{0}1110100 \cdots$

$f(2): 00\boxed{0}0000000 \cdots$

$f(3): 010\boxed{1}01010 \cdots$

Def $D: \mathbb{Z}_{\geq 0} \to \{0, 1\}$ defined by $D(n) = 1 - f(n)(n)$

Claim: D is not on the list

Pf: Suppose it is $-$ $D = f(m)$ for some $m \in \mathbb{Z}_{\geq 0}$

so $D(m) = f(m)(m)$

$D(m) = 1 - f(m)(m)$ X

Collary $\mathbb{R}$ is uncountable

Pf: Let S be set of decimals with only 0's and 1's

$0.1001000111 \cdots$ $\quad$ $S \sim \{0, 1\}^{\mathbb{Z}_{\geq 0}}$

$0.11101111 \cdots$ $\quad$ so S is uncountable

But $S \subset \mathbb{R}$ if $\mathbb{R}$ is countable, S would be countable

In fact, $\mathbb{R} \sim \{0, 1\}^{\mathbb{Z}_{\geq 0}}$

The equivalence class of $\mathbb{Z}_{\geq 0}$ is called $\aleph_0$

... $\{0,1\}^{\mathbb{Z}_{\geq 0}}$ is $\aleph_1$ continuum

$\{0,1\}^{(\{0,1\}^{\mathbb{Z}_{\geq 0}})}$ $\aleph_2$

The set of all finite-length English texts is countable

$\Rightarrow$ There exists real numbers which cannot be described

SOME HAVE SUGGESTED: (Intuitionism, Constructionism, Brouwer, etc.)

$\mathbb{R}$ does not exist