MATH 521 Lecture 9 R.T.

VECTOR SPACES

A vector space over a field k (a k-vector space) is a set V, whose elements are called <u>vectors</u>, with operations $+$ and $\cdot$

$+$: addition

$\cdot$: scalar multiplication

i.e. given $v, w \in V$ $\quad v+w \in V$

given $v \in V$, $\lambda \in k$, $\lambda v \in V$

satisfying many rules, e.g. $v+w = w+v$ $\quad \lambda(v+w) = \lambda v + \lambda w$ $\quad \cdots$

<u>E.G.</u>

- k is a k-vector space
- k^n is a k-vector space $\quad v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ or $(x_1, \ldots, x_n)$

  (in particular, $\mathbb{R}^n$ is an $\mathbb{R}$-vector space)
- V = the set of all functions $f: \mathbb{R} \to \mathbb{R}$

  $\cup$
- $V_{cts} = \cdots$ <u>continuous</u> functions $f: \mathbb{R} \to \mathbb{R}$ is a <u>subspace</u> of V

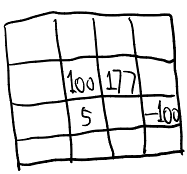
  (need sum of two continuous functions is continuous)
- V_b = the set of all <u>bounded</u> $f: \mathbb{R} \to \mathbb{R}$
- $\{$function f: States $\to \mathbb{R}\}$
- $\mathbb{R}$ is a vector space over $\mathbb{Q}$

NORM

We want a way of measuring how "large" a vector in V is

<u>E.G.</u> V be the space of functions on a 640×640 grid $\qquad \mathbb{R}^{409600}$

$\mathbb{R}^{640 \times 640}$



$f(x,y)$ specifies the brightness of pixel (x,y)

So each vector in V is an <u>image</u>

Suppose you have stored images $V_1, V_2, V_3, \ldots, V_N$

W a new image

You believe w is a match for some V_i, but which one?

Which V_i is closet to w?

When is $V_i - W$ small?

What could we mean by the <u>size</u> of a vector $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$?

$\|(x_1, \ldots, x_n)\|_2 \quad (x_1^2 + \cdots + x_n^2)^{1/2}$ $\quad L^2$ norm

$\|(x_1, \ldots, x_n)\|_\infty \quad \max_i |x_i|$ $\quad L^\infty$ norm

$\|(x_1, \ldots, x_n)\| \quad |x_1| + |x_2| + \cdots + |x_n|$ $\quad L^1$ norm

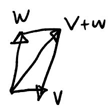
$\|(x_1, \ldots, x_n)\|_p \quad (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ $\quad L^p$ norm

<u>Def</u> V vector space / $\mathbb{R}$

A <u>norm</u> on V is a function $\|\cdot\| : V \to \mathbb{R}$ s.t.

- $\|v\| \geq 0 \ \forall v \in V$, and $\|v\| = 0$ iff v_0
- $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in \mathbb{R}$, $v \in V$
- $\|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in V$

e.g. for the L^2 norm, says:



What about L^∞ norm on $\mathbb{R}^n$

$v = (x_1, \ldots, x_n) \quad w = (y_1, \ldots, y_n)$

$v + w = (x_1 + y_1, \ldots, x_n + y_n)$

To show,

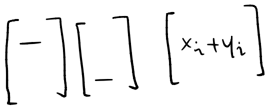
$$\max_i |x_i + y_i| \leq \max_i |x_i| + \max_i |y_i|$$

Let j in $[1, \ldots, n]$ be the index s.t.

$$|x_j + y_j| = \max_i |x_i + y_i|$$

$$j = \operatorname{argmax}_i |x_i + y_i|$$

$|v+w|_\infty = \max_i |x_i + y_i| = |x_j + y_j| \leq |x_j| + |y_j|$



$\leq \max_i |x_i| + \max_i |y_i|$

$= |v|_\infty + |w|_\infty$

RK: the set of functions $f: \mathbb{R} \to \mathbb{R}$ bounded between -1 and 1 are <u>not</u> a vector space

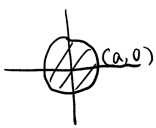
$\sin \in V$ but their sum $2\sin$ is <u>not</u> in V
$\sin \in V$

V vector space
$\|\cdot\|$ a norm on V

<u>Def</u> The <u>norm ball</u> $B_{V, \|\cdot\|}(a)$ is the set $\{v \in V : \|v\| \leq a\}$

Norm balls in $\mathbb{R}^2$

$B_{\mathbb{R}^2, |\cdot|_2}(a)$

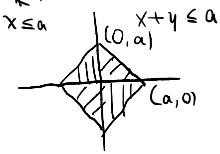


$(x^2+y^2)^{1/2} \leq a$

$x^2 + y^2 \leq a^2$

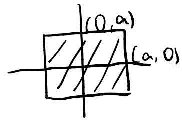
$B_{\mathbb{R}^2, |\cdot|_1}(a)$

$y - x \leq a$ $\quad x + y \leq a$



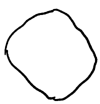
$|x| + |y| \leq a$

$B_{\mathbb{R}^2, |\cdot|_\infty}(a)$

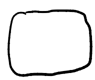


$\max(|x|, |y|) \leq a$

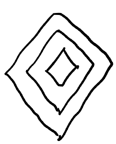
$\mathbb{R}, |\cdot|_{1.5}$



$\mathbb{R}, |\cdot|_4$

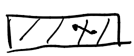


L_1 balls for different a



<u>Def</u> We say a subset S of a vector space V is <u>convex</u> if, for every v, w in S, the line segment $\overline{vw}$ is continued in S

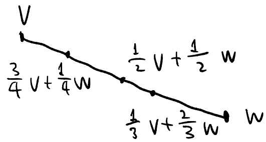
non-convex



convex

Fact: Norm balls are always convex

$\|\lambda v\| = |\lambda|\,\|v\|$



every point on this line segment

$\lambda v + (1-\lambda) w \quad \lambda \in [0,1]$

Pf that norm balls are convex:

Suppose $v, w \in B_{V, \|\cdot\|}(a)$. To show:

$\|\lambda v + (1-\lambda) w\| \leq a$ (given $\|v\|, \|w\| \leq a$)

$$\begin{aligned} \|\lambda v + (1-\lambda) w\| &\leq \|\lambda v\| + \|(1-\lambda) w\| \\ &= |\lambda|\,\|v\| + |1-\lambda|\,\|w\| \\ &\not\equiv \lambda \|v\| + (1-\lambda)\|w\| \\ &\leq \lambda a + (1-\lambda) a = a \quad \checkmark \end{aligned}$$

Are there good norms on space of $f: \mathbb{R} \to \mathbb{R}$

... continuous functions?

... bounded functions?

Def Let V_b space of bounded functions $f: \mathbb{R} \to \mathbb{R}$

The sup norm on V_b is defined by

$$\|f\|_{sup} = \sup_x |f(x)|$$

Why does this exist?

$E_f = \{|fx|\}_{x \in \mathbb{R}}$

E_f is nonempty $|f(7)| \in E_f$

E_f is bounded above because f is bounded

$\Rightarrow$ By LUBP, E_f has a supremum

To show this is a norm, need to show

$$\|f+g\|_{sup} \leq \|f\|_{sup} + \|g\|_{sup}$$

$\mathbb{R}^n = \{(x_1, \ldots, x_n)\}$

$V = \{(f(0), f(1), f(2), f(1.79), f(\pi), \ldots\}$

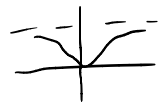
What I'd like to say is:

let x_0 be that real # such that

$|f(x_0)| = \sup_x |f(x)|$

$=$ "$\arg\sup |f|$"

$f(x) = 1 - \frac{1}{(x^2+1)}$ $\qquad \|f\|_{\sup} = 1$



A natural question:

is there some analogue of L^2 norm for functions? Yes

See <u>inner product spaces</u>