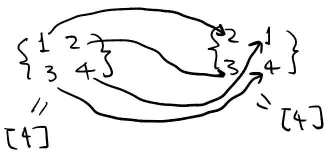
MATH 541 Lecture 2 Dihedral Groups

Last time: $[n] = \{1, 2, \dots, n\}$

$S_n = \mathrm{Aut}([n]) = \{f : [n] \to [n] \mid f \text{ is a bijection}\}$

Symmetry group of n elements

"$\underbrace{\text{auto}}_{\text{self}}\underbrace{\text{morphism}}_{\text{map}}$" = an operation / transformation on an object, after which the object coincides w/ itself



$(1, 2, 3, 4)$ cycle representation of an element in $\mathrm{Aut}([n])$

$1 \mapsto 2 \quad 2 \mapsto 1 \quad 3 \mapsto 4 \quad 4 \mapsto 3 \qquad (1,2)(3,4)$

$(G, *)$ $\quad *$ = binary operation

Usually, we just write ab for $a * b$

1 for e (identity)

0 for e (if G is abelian)

e.g. $(\mathbb{Z}, +)$

<u>Def</u> The order $|g|$ of an element $g \in G$ is the smallest $n \in \mathbb{Z}_{>0}$ s.t. $g^n = 1$

$g^n = \underbrace{g \cdots g}_{n \text{ times}}$

$g^0 \overset{\text{def}}{=} 1 \qquad g^{-n} = (g^{-1})^n$

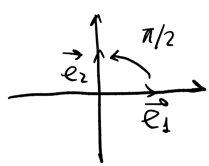
$|(1, 2, 3, 4)| = 4$

$\pi/2$ on $\mathbb{R}^2 \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_{2\times 2}^{\times} = \{A \in M_{2\times 2} \mid \det(A) \neq 0\}$

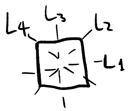
$\vec{e}_1, \vec{e}_2$ standard basis $\mathbb{R}^2$

$\vec{e}_1 \mapsto \vec{e}_2$

$\vec{e}_2 \mapsto -\vec{e}_1$



Question: In how many ways is a square symmetric to itself?



(1) Rotate by $\pi/2$ n times denote as r_n

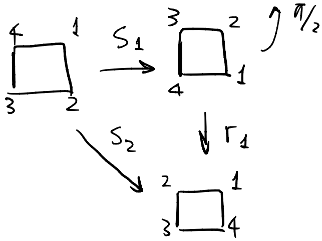
(2) Reflect across any axis m denote as s_m

symmetry = automorphism $\quad r_5 = r_1 \quad r_1^{-1} = r_3 \quad |r_1| = 4$

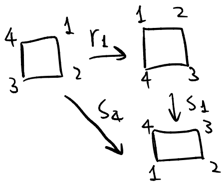
$|s_m| = 2, \ \forall m$

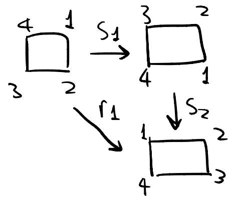
$$\mathrm{Aut}(\square) = \left\{ \begin{matrix} r_1 & r_2 & r_3 & r_4 \\ s_1 & s_2 & s_3 & s_4 \end{matrix} \right\}$$

$r_1 s_1 = s_2$



Compute $\quad s_1 r_1 = s_4 \qquad s_2 s_1 = r_1$





Analogue. $\quad \varphi : V \to W \quad$ linear transformation btw vector spaces

$\varphi(\vec{v}) \quad$ In practice, pick a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for V

Only need to specify $\varphi(\vec{v})$

Set $r_1 = r$, $s_1 = s$

$$\begin{matrix} & r & r^2 & r^3 & r^4 = 1 \\ & \| & \| & \| & \| \\ \mathrm{Aut}(\square) = & \{ r_1 & r_2 & r_3 & r_4 \\ & s_1 & s_2 & s_3 & s_4 \} \\ & \| & \| & \| & \| \\ & s & rs & r^2 s & sr = r^3 s = r^{-1} s \end{matrix}$$

$$\mathrm{Aut}(\square) = \langle \underbrace{r, s}_{\text{generators}} \mid \underbrace{r^4 = 1, \ s^2 = 1, \ sr = r^{-1} s}_{\text{relations}} \rangle$$

$$s_1 r_4 r_3 s_2 s_3 r_2 = s r^{-1} r s r^2 r^{-2} s$$
$$= s$$

$$D_{2n} = \langle r, s \mid r^n = 1,\ s^2 = 1,\ rs = sr^{-1} \rangle$$

number of elements $= |D_{2n}| = 2n$