MATH 541 Lecture 04 1 Feb 2023

Group Homomorphism (morphisms)

$\uparrow$

universal term for maps btw structured sets

(between)

<u>Def</u> Let G, H be groups

A morphism $\varphi : G \to H$ is a map of sets

s.t. $\varphi(x) *_H \varphi(y) = \varphi(x *_G y)$, $\forall x, y \in G$

Recall: V, W are vector spaces Then a morphism (aka linear transformation)

$\varphi : V \to W$ is a map of sets s.t. $\varphi(\vec{x}+\vec{y}) = \varphi(\vec{x}) + \varphi(\vec{y})$, $\forall \vec{x}, \vec{y} \in V$

and $\varphi(c\vec{x}) = c\,\varphi(\vec{x})$, $\forall \vec{x} \in V$ $\varphi(\vec{0}) = \vec{0}$

Similarly, $\varphi(e_G) = e_H$ for a group morphism

$\varphi(x * e_G) = \varphi(x) * \varphi(e_G)$

$\parallel$

$\varphi(x)$

Set $y = \varphi(x)$

$y^{-1} y \varphi(e_G) = y^{-1} y \Rightarrow \varphi(e_G) = e + 1$

Rmk: V = vector space $(V, +)$ = abelian group

$\varphi : V \to W$ is in particular a morphism

linear transformation $(V,+) \to (W, +)$

Similarly, $\varphi(x^{-1}) = \varphi(x)^{-1}$

$\varphi(\underbrace{x \cdot x^{-1}}_{e_G}) = \varphi(x)\, \varphi(x^{-1})$

<u>Lemma</u> Say $\varphi : G \hookrightarrow H$ is an injective group morphism $\hookrightarrow$ injective

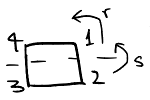
Then $x_1 \cdot x_2 \cdot \dots \cdot x_n = 1$ in G holds

if and only if $\underbrace{\varphi(x_1) \cdot \dots \cdot \varphi(x_n)}_{\varphi(x_1 \cdot \dots \cdot x_n)} = 1$ in H "only if" – def of morphism

"if" – $\varphi(x_1 \cdots\cdots x_n) = 1 \qquad \varphi(1) = 1 \qquad$ injective

$\Rightarrow x_1 \cdots\cdots x_n = 1$

Revisit the dihedral group

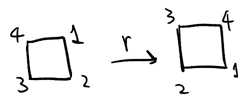


$D_8 = \mathrm{Aut}(\square) \overset{\varphi}{\hookrightarrow} \mathrm{Aut}([4]) = S_4$

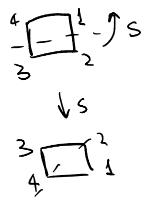
$[4] =$ set of vertices

An automorphism of $\square$ is completely determined by its induced auto on the set of vertices

Example $\quad r$ induces a bijection $[4] \to [4]$

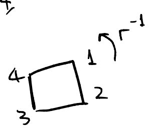


$\varphi(r) = (1\ 2\ 3\ 4) \in S_4$



$\varphi(s) = (1\ 2)(3\ 4)$

$\varphi(s') = (2\ 4)$

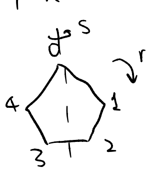


$\varphi(r^{-1}) = (4321)$

$\varphi(g) = \varphi(h) \Rightarrow g = h \Rightarrow \varphi$ is injective

$(3\ \ 4) \notin \underset{\substack{\uparrow \\ \text{image}}}{\mathrm{im}}(\varphi) \qquad (1\ \ 2\ \ 3) \notin \mathrm{im}(\varphi)$

Say $n = 5$



$\varphi(r) = (0\ \ 1\ \ 2\ \ 3\ \ 4)$

$\varphi(s) = (1\ \ 4)(2\ \ 3)$

Verify the relation $\quad rs = sr^{-1}$

Suffices to check $\quad \varphi(r)\varphi(s) = \varphi(s)\varphi(r^{-1})$

b/c φ is injective

$$\varphi(r)\varphi(s) = (0\ 1\ 2\ 3\ 4)(1\ 4)(2\ 3)$$
$$= (0\ 1)(2\ 4)$$

$$\varphi(s)\varphi(r^{-1}) = (1\ 4)(2\ 3)(4\ 3\ 2\ 1\ 0)$$
$$= (0\ 1)(4\ 2)$$

$0 \to 0$	$0 \to 0$	$0 \to 4$
$1 \to 4$	$1 \to 1$	$1 \to 0$
$2 \to 2$	$2 \to 3$	$2 \to 1$
$3 \to 3$	$3 \to 2$	$3 \to 2$
$4 \to 1$	$4 \to 4$	$4 \to 3$

Works if I replace 5 by any odd number