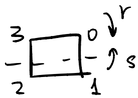
MATH 541 Lecture 05 3 feb 2023

Last time: $D_{2n} \xrightarrow{\varphi} \mathrm{Aut}(X)$ X = set of vertices

Ex: $n=4$



$\varphi(r) = (0\ 1\ 2\ 3)$

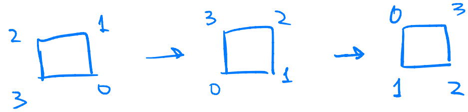
$\varphi(s) = (0\ 1)(2\ 3)$

$\varphi(r^2 s) = (0\ 3)\ (1\ 2)$

$= (0\ 1\ 2\ 3)(0\ 1\ 2\ 3)\ (0\ 1)\ (2\ 3)$

$X = [4] = \{0,1,2,3\}$

$\mathrm{Aut}(X) = S_4$



<u>Def</u> G = group X = set

An action of G on X is a group homomorphism

$G \to \mathrm{Aut}(X) \overset{\text{def}}{=} \{f : X \to X \mid f \text{ bijection}\}$ group law = composition

Slogan: A group is like a man and he should be understood through his actions

Consider $\chi : D_{2n} \to \mathbb{Z}/2\mathbb{Z}$

As a function of <u>sets</u>

$\chi: A = \{1, r, \dots, r^{n-1}\} \to \bar{0}$ $B = \{rs, r^2 s, \dots, r^{n-1}s\} \to \bar{1}$

e.g. $r^n s = s r^{-n}$

Check: χ is a group morphism

Need: $\chi(xy) = \chi(x) + \chi(y)$

$y\in$ \ $x\in$	A	B
A	1	2
B	3	4

① $x = r^i \quad y = r^j$ for some i, j

$\psi(xy) = \psi(r^{i+j}) = \bar{0}$

$\psi(x) + \psi(y) = \bar{0} + \bar{0} = \bar{0}$ ✓

② $x = r^i s \quad y = r^j \qquad \psi(xy) = \psi(r^i s r^j) = \psi(r^{i-j} s) = \bar{1}$

$\psi(x) + \psi(y) = \bar{1} + \bar{0} = \bar{1}$

④ $x = r^i s \qquad y = r^j s$

$\psi(xy) = \psi(r^i s r^j s) = \psi(r^{i-j}) = \bar{0}$

$\psi(x) + \psi(y) = \bar{1} + \bar{1} = \bar{0}$

In general if G is given by generators and relations, to check whether $\psi: G \to H$ (H = group) is a group morphism, it suffices to check that ψ represents the relations

<u>Ex</u> For $\psi: D_{2n} \to \mathbb{Z}/2\mathbb{Z}$ as above, only need to check $\psi(r^n) = \bar{0}$, $\psi(s^2) = \bar{0}$, and $\psi(rsrs) = \bar{0}$

$D_{2n} = \langle r, s \mid r^n = 1,\ s^2 = 1,\ rsrs = 1 \rangle$

X = set

2^X = power set of X

= set of all subsets of X

There is a natural action of $\mathrm{Aut}(X)$ on 2^X

$f \in \mathrm{Aut}(X) \xrightarrow{\varphi} \mathrm{Aut}(2^X)$

$f: X \to X$ bijection

$\varphi(f) \in \mathrm{Aut}(2^X)$

$A \subseteq X \qquad \varphi(f)(A) = \{f(a) \mid a \in A\} \subseteq X$

$\varphi(f) \in \mathrm{Aut}(2^X)$

$\varphi(f): 2^X \to 2^X$

$A \subseteq X$

$\varphi(f)(A) \in 2^X$

Say $X = \{0, 1, 2, 3\}$ $\quad f = (0\ 1\ 3)$

$A \in 2^X$, say $A = \{2, 3, 1\} \subseteq X$

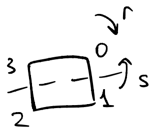
$\varphi(f)(A) = \{2, 0, 3\} \subseteq X$

$n = 4$

$$A = \{2, 3, 1\} \xrightarrow{f} \{2, 0, 3\}$$

$$D_8 \to S_4 = \mathrm{Aut}(X) \to \mathrm{Aut}(2^X)$$

X = set of vertices



Ex $\quad A = \{0, 2\}$

$$r \cdot A \overset{\text{notation}}{=} \partial(r)(A)$$

$r \cdot A = \{1, 3\} \quad s \cdot A = \{1, 3\}$

Claim: $\forall g \in D_{2n}$, either $\begin{cases} g \cdot A = A \\ g \cdot B = B \end{cases}$ or $\begin{cases} g \cdot A = B \\ g \cdot B = B \end{cases}$

Example: $1 \cdot A = A \qquad r^2 \cdot A = A$

Then I define $\pi : D_{2n} \to \mathrm{Aut}(\{A, B\}) \cong \mathbb{Z}/2\mathbb{Z}$

$\pi \neq \psi$ because $\psi(r) = \bar{0} \quad \pi(r) = \bar{1}$