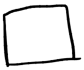
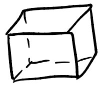
Lecture 6 Symmetry groups and cubes

Question. What is the symmetry group of a cube?

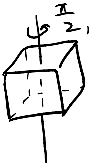


D_8



Aut group = ?

trivial = identity



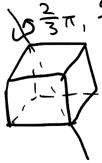
$\frac{\pi}{2}, \pi, \frac{3}{2}\pi$

6 faces

Type I rotations

$9 = \underset{\uparrow}{3} \times 3$ nontrivial rotation

\# axis of this shape

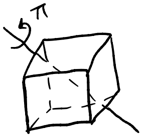


$\frac{2}{3}\pi, \frac{4}{3}\pi$

Type II rotations

$8 = \underset{\uparrow}{4} \times 2$ nontrivial ones

\# axis of this type



π

Type III rotations

$\underset{\uparrow}{6} \times 1$

\# axis of this type

In total, we have $9 + 8 + 6 + \underset{\uparrow}{1} = 24$ elements in the

identity element

G = symmetry group of a cube

It turns out that every element in G can be obtained by composing Type III rotations!

Symmetry group S_n revisited

$S_n = \mathrm{Aut}([n])$ $\quad [n] = \{0, 1, 2, \ldots, n-1\}$

Def A transposition in S_n is an element of the form $\underbrace{(a_1, a_2)}_{\text{cycle representation}}$, for $a_1, a_2 \in [n]$

e.g. $(0,1)$, $(2,3)$, $(1,3)$ are transpositions in S_3

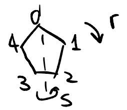
Thm S_n is generated by transpositions, i.e., $\forall g \in S_n$, $\exists$ an expression $g = g_1 \cdots g_m$ for some m and g_i is a transposition for every i

E.g. Say we consider $(0, 1, 2, \ldots, n-1) \in S_n$

How to decompose it into transpositions?

Consider the dihedral group $D_{2n} \subseteq S_n$

e.g. $n=5$



Then $(0, 1, 2, 3, 4) = r$

We already know that the reflections are products of transpositions

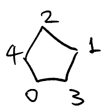
s, rs are both reflections

$r = (rs)(s)$

$s = (1\ 4)(2\ 3)$

$rs = (1\ 0)(2\ 4)$

What about $(2\ 1\ 3\ 0\ 4)$



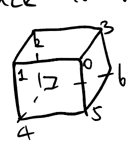
same logic applies

But we know that all elements in S_n can be expressed as a product of cycles, and a cycle can be decomposed into a product of transpositions

Exercise: Decompose $(2\ 1\ 3)(3\ 4\ 5) \in S_6$ into a product of transpositions

Exercise: Use the above to write down a <u>formal</u> proof of the theorem

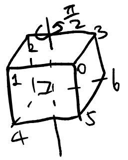
Back to the cube



$\{0, 7\}$, $\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$

A B C D

action on $\{A, B, C, D\}$



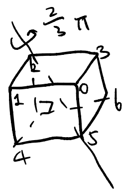
$(A\ D\ C\ B)$

$A\ \{0,7\} \mapsto D\ \{3,4\}$

$C\ \{2,5\} \mapsto B\ \{1,6\}$

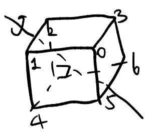
$B\ \{1,6\} \mapsto A\ \{0,7\}$

$D\ \{3,4\} \mapsto C\ \{2,5\}$



$\{2,5\} \to \{2,5\}$

Left as an exercise



$(B\ C)$

$\overset{B}{\{1,6\}} \to \overset{C}{\{2,5\}}$

All other pairs are fixed

Similarly, you can realize all other transpositions by reflecting at suitable axis of type IV

$$\#\text{ transpositions in } S_4 = \binom{4}{2} = \frac{4\times 3}{1\times 2} = 6$$

Now I consider my action of G = symmetry group of cube on the set $\{A, B, C, D\}$ of pairs

$$\mathrm{Aut}(\{A, B, C, D\}) \overset{\sim}{=} S_4$$

isomorphism not unique

$$G \to \mathrm{Aut}(\{A, B, C, D\})$$

Hw: A group morphism is injective if and only if the kernel is $\{1\}$

$\ker(\varphi) = \{g \in G \mid \varphi(g) = 1\} = \{1\}$ in our case

$\Rightarrow \varphi$ is injective

Is φ surjective? Answer = Yes!

B/C $\mathrm{Aut}(\{A, B, C, D\}) \cong S_4$ is generated by transpositions, which all lie in the image of φ

Exercise: Suppose $\varphi : G \to H$ for general groups G and H

If $\mathrm{im}(\varphi)$ contains a set of generators of H, then φ is surjective

$Y \subseteq H$ is a set of generators if $\forall h \in H$ is of the form
subset
$h_1, \ldots, h_m$ for some m and $h_i \in Y$

Conclusion: Symmetry group of a cube is isomorphic to S_4!