Lecture 7 Abstract Nonsense

Basics of Set Theory

Say $f: A \to B$ is a map of sets. Then f is surjective iff (if and only if) it has the following property:

For every $g_1, g_2: B \to C$ (C = some other set) s.t. $g_1 \circ f = g_2 \circ f$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g_1 \circ f"] \arrow[rr, bend right, "g_2 \circ f"'] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

Take any $b \in B$ want to show $g_1(b) = g_2(b)$

We know $\exists a \in A$ s.t. $f(a) = b$

$$\begin{array}{ccc} g_1(b) & = & (g_1 \circ f)(a) \\ \| & & \| \\ g_2(b) & = & (g_2 \circ f)(b) \end{array}$$

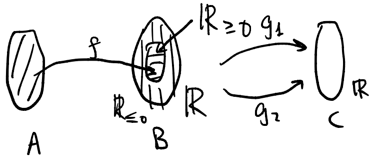
converse = exercise

Then $g_1 = g_2$

Nonexample: Concider $f: \mathbb{R} \to \mathbb{R}$ defined by $x \to x^2$

Let $g_1, g_2: \mathbb{R} \to \mathbb{R}$ be $g_1(x) = x$, and $g_2(x) = |x|$

Then $g_1 \circ f = g_2 \circ f$, but $g_1 \neq g_2$



f "does not know" about $\mathbb{R}_{\leq 0}$ in B

$g_1 = g_2$ over $\mathbb{R}_{\geq 0}$

$g_1 \neq g_2$ over $\mathbb{R}_{<0}$

$f: A \to B$ is injective $\Leftrightarrow$ $\left(\begin{array}{l} \forall g_1, g_2: C \to A \text{ s.t.} \\ \quad f \circ g_1 = f \circ g_2 \\ \text{then } g_1 = g_2 \end{array} \right)$

$$\begin{tikzcd} C \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[rr, bend left, "f \circ g_1"] \arrow[rr, bend right, "f \circ g_2"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Left as Exercise

Factorization of maps

Let $f: A \to B$ be maps of sets
$g: A \to C$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

We say that f factors through g if $\exists h: C \to B$
s.t. $f = h \circ g$ (i.e., $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[ur, "h"'] & \end{tikzcd}$ commutes)

If g is surjective, then h has to be unique if it exists

(if $h': C \to B$ is another map s.t.
$f = h' \circ g$, then $h = h'$)

Equivalence Relations.

Recall that X = set 2^X = set of all subsets of X

Given $\lambda \in 2^X$, write X_λ for the corresponding subset of X

<u>Def</u> A binary relation $\sim$ on X is a subset $R \subseteq X \times X$ s.t.

$x \sim y$ iff $(x, y) \in R$

Eg. $R \subseteq \mathbb{R} \times \mathbb{R}$ be defined by $R = \{(x, y) \mid \exists r \in \mathbb{R}, \text{ s.t. } y - x = r^2\}$

Then $x \sim y \Leftrightarrow x \leq y$ i.e., $\sim$ = "$\leq$"

Similarly, $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be defined by

$$R = \{(a, b) \mid \exists k \in \mathbb{Z} \text{ s.t. } b - a = nk\}$$

Then $x \sim y \Leftrightarrow x \equiv y \mod n$

We say that $\sim$ is an equivalence relation if it is

(a) reflexive $x \sim x$ (i.e., $(x, x) \in R$)

(b) symmetric $x \sim y \Leftrightarrow y \sim x$

(c) transitive $x \sim y$ $y \sim z \Rightarrow x \sim z$

Non-examples: $<$ is not reflexive but $\leq$ is

$<$ and $\leq$ are both transitive

$(x \leq y \quad y \leq z \Rightarrow x \leq z)$

but not symmetric (if $x < y$ then $y < x$ is false)

Example: $\equiv \bmod n$ is an equivalence relation!

Clearly $\forall x \in \mathbb{Z} \quad x \equiv x \bmod n$

$\forall x, y \in \mathbb{Z} \quad \left(x \equiv y \bmod n \Leftrightarrow y \equiv x \bmod n \right)$

$\exists k \in \mathbb{Z}$ s.t. $y - x = kn \qquad x - y = (-k) \cdot n \quad -k \in \mathbb{Z}$

Transitive. $x \equiv y \bmod n \quad y \equiv z \bmod n \Rightarrow x \equiv z \bmod n$

$y - x = kn$, $k \in \mathbb{Z}$ $\qquad z - y = \ell n$, $\ell \in \mathbb{Z}$ $\qquad$ b/c $z - x = (k + \ell) n$, $k + \ell \in \mathbb{Z}$

Suppose that if define $(n \in \mathbb{N})$ $R \subseteq \mathbb{Z} \times \mathbb{Z}$

by $\{(a, b) \mid b - a = kn \text{ for some } k \in \mathbb{N}\}$

Then the resulting $\sim$ is not symmetric

Partition A partition of a set X is given by a subset

$$I \subseteq 2^X \text{ s.t. } \boxed{\begin{cases} \bigcup_{i \in I} X_i = X \\ \forall i, j \in I,\ i \neq j,\ X_i \cap X_j = \emptyset \end{cases}}$$

$$\Updownarrow$$

$$\forall x \in X,\ \exists!\, i \in I, \text{ s.t. } x \in X_i$$

When satisfied, we have a well-defined map $X \xrightarrow{\pi} I$

s.t. $\forall x \in X$, $\pi(x) \in 2^X$ is the unique element s.t. $x \in X_{\pi(x)}$

E.g., For $b \in \mathbb{Z}$, write $b + \mathbb{Z} = \{a \in \mathbb{Z} \mid a - b = kn \text{ for some } k \in \mathbb{Z}\}$

Then $I = \{n\mathbb{Z}, 1 + n\mathbb{Z}, \ldots, (n-1) + n\mathbb{Z}\} \subseteq 2^{\mathbb{Z}}$ is a partition of $\mathbb{Z}$

For a set X, giving an equivalence relation is the same thing as giving a partition

Suppose we are given an equivalence relation $\sim$
$\Rightarrow I \subseteq 2^X \quad I = \{\lambda \in 2^X \mid X_\lambda = \{y \in X \mid y \sim x\}$ for some $x \in X\}$

$x \sim y \Leftrightarrow \pi(x) = \pi(y) \quad \Leftarrow$ Suppose we are given a partition $I \subseteq 2^X$

Revisit the statement that $+$ on $\mathbb{Z}$ gives rise to a well-defined "$+$" on $\mathbb{Z}/n\mathbb{Z}$

$\mathbb{Z}/n\mathbb{Z} \subseteq 2^{\mathbb{Z}}$

$x \mapsto x + n\mathbb{Z}$ (or $\bar{x}$)

$(x, y) \mapsto x + y$

$$\begin{tikzcd} \mathbb{Z} \times \mathbb{Z} \arrow[r, "+"] \arrow[d, "\pi \times \pi"'] & \mathbb{Z} \arrow[d, "\pi"] \\ \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \arrow[r, "``+\text{''}"] & \mathbb{Z}/n\mathbb{Z} \end{tikzcd}$$

$\exists$ "$+$" s.t. the diagram commutes
$\downarrow \quad \pi \circ + = \text{"}+\text{"} \circ (\pi \times \pi)$
is in particular a binary operation on $\mathbb{Z}/n\mathbb{Z}$

More generally. suppose that $(G, *)$ is a group

$\sim$ is an equivalence relation on G and let G/N be the resulting partition of G.

If $\exists$ "$*$" s.t.

$$\begin{tikzcd} G \times G \arrow[r, "*"] \arrow[d, "\pi \times \pi"'] & G \arrow[d] \\ G/N \times G/N \arrow[r, "``*\text{''}"] & G/N \end{tikzcd}$$

commutes

then "$*$" defines a group structure on G/N