CS 540 Introduction to Artificial Intelligence
Games I
University of Wisconsin-Madison

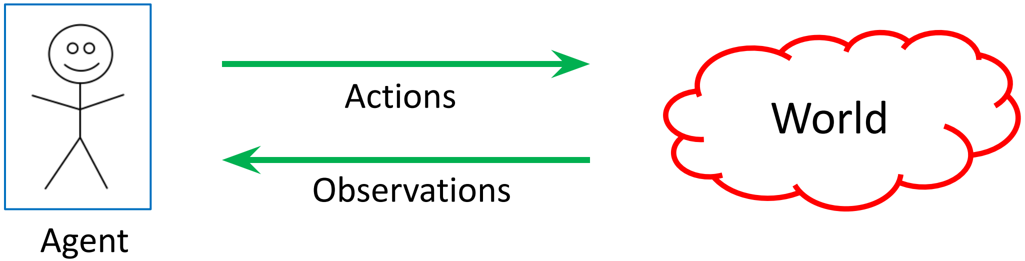
Spring 2022

Outline

- Introduction to game theory
  - Properties of games, mathematical formulation
- Simultaneous-Move Games
  - Normal form, strategies, dominance, Nash equilibrium

More General Model

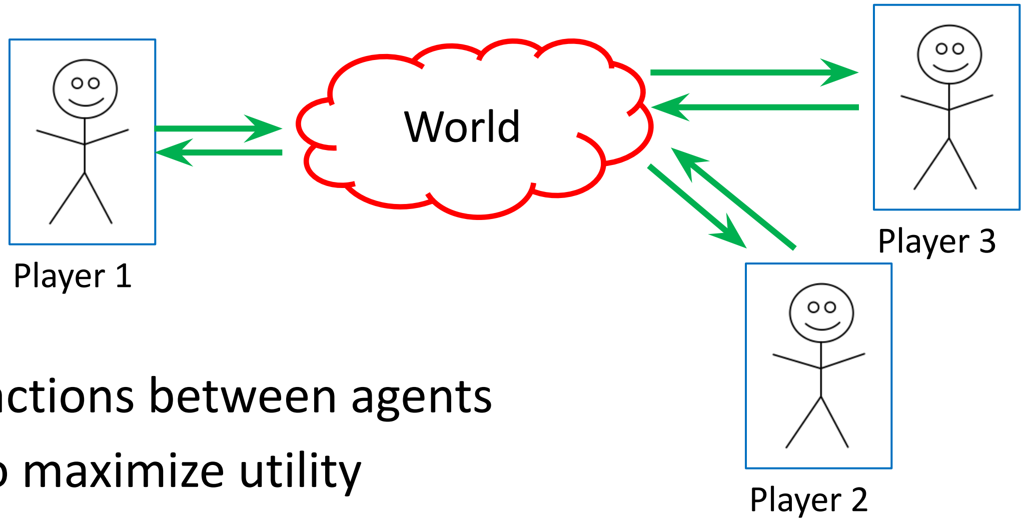
Suppose we have an **agent** **interacting** with the **world**



- Agent receives a reward based on state of the world
  - **Goal**: maximize reward / utility **($$$)**
  - Note: now **data** consists of actions & observations
  - Setup for decision theory, reinforcement learning, planning

Games: Multiple Agents

Games setup: **multiple** agents



- Now: interactions between agents
- Still want to maximize utility
- **Strategic** decision making.

Modeling Games: Properties

Let's work through **properties** of games

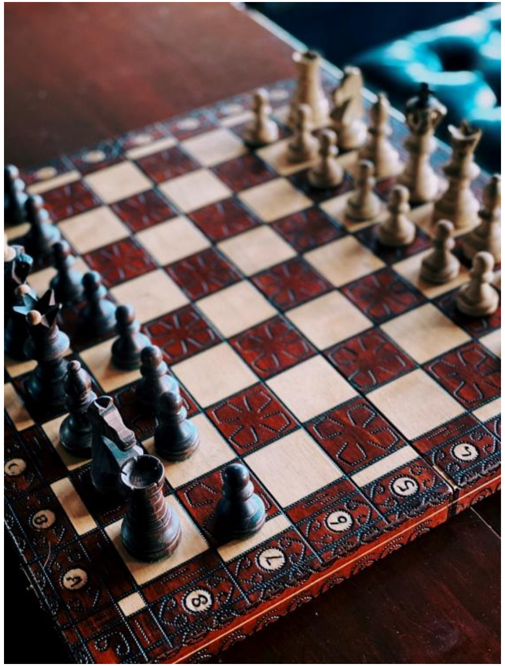
- **Number** of agents/players
- Action space: finite or infinite
- **Deterministic** or **random**
- Zero-sum or general-sum
- **Sequential** or **simultaneous moves**

Wiki

Property 1: **Number** of players

Pretty clear idea: 1 or more players

- Usually interested in ≥ 2 players
- Typically a finite number of players



Property 2: Action Space

Finite or infinite

- Rock-paper-scissors
- Tennis

Property 3: **Deterministic** or **Random**

- Is there **chance** in the game?
  - Poker
  - Scrabble
  - Chess

Property 4: **Sum of payoff**

- Zero sum: one player's win is the other's loss
  - Pure competition. E.g. rock-paper-scissors

- General sum
  - Example: prisoner's dilemma

Property 5: **Sequential** or **Simultaneous Moves**

- Simultaneous: all players take action at the same time
- Sequential: take turns (but payoff only revealed at end of game)

Normal Form Game

Mathematical description of simultaneous games.

- n players $\{1,2,\ldots,n\}$
- Player i strategy a_i from A_i.
- Strategy profile: $a = (a_1, a_2, \ldots, a_n)$
- Player i gets rewards $u_i(a)$
  - **Note**: reward depends on other players!

- We consider the simple case where all reward functions are common knowledge.

Example of Normal Form Game

Ex: Prisoner's Dilemma

Player 2 / Player 1	*Stay silent*	*Betray*
Stay silent	−1, −1	−3, 0
Betray	0, −3	−2, −2

- 2 players, 2 actions: yields 2x2 payoff matrix
- Strategy set: {Stay silent, betray}

Strictly Dominant Strategies

Let's analyze such games. Some strategies are better

- Strictly dominant strategy: if a_i strictly better than a_i' *regardless* of what other players do, a_i is **strictly dominant**
- I.e., $u_i(a_i, a_{-i}) > u\_i(b, a_{-i}), \forall b \neq a_i, \forall a_{-i}$

All of the other entries of *a* excluding *i*

- Doesn't always exist!

Strictly Dominant Strategies Example

Back to Prisoner's Dilemma

- Examine all the entries: betray strictly dominates
- Check:

Player 2 Player 1	*Stay silent*	*Betray*
Stay silent	−1, −1	−3, 0
Betray	0, −3	−2, −2

Dominant Strategy Equilibrium

$a*$ is a (strictly) dominant strategy equilibrium, if all players have a strictly dominant strategy a_i^*

- Rational players will play at DSE, if one exists.

Player 2 Player 1	*Stay silent*	*Betray*
Stay silent	−1, −1	−3, 0
Betray	0, −3	−2, −2

Dominant Strategy: Absolute Best Responses

Player i's best response to a_{-i}: $BR(a_{-i}) = \arg\max_a u\_i(a, a_{-i})$

BR(player2=silent)=betray

BR(player2=betray)=betray

Player 2 Player 1	*Stay silent*	*Betray*
Stay silent	−1, −1	−3, 0
Betray	0, −3	−2, −2

a_i^* is the dominant strategy for player i, if

$a_i^* = BR(a_{-i}), \forall\, a_{-i}$

Dominant Strategy Equilibrium

DSE does not always exist.

Player 2 Player 1	*L*	*R*
T	2, 1	0, 0
B	0, 0	1, 2

Nash Equilibrium

$a*$ is a Nash equilibrium if no player has an incentive to **unilaterally deviate**

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i$$

Player 2 / Player 1	*L*	*R*
T	2, 1	0, 0
B	0, 0	1, 2

Nash Equilibrium : Best Response to Each Other

$a*$ is a Nash equilibrium:

$$\forall i, \forall b \in A_i: u_i(a_i^*, a_{-i}^*) \geq u_i(b, a_{-i}^*)$$

(no player has an incentive to **unilaterally deviate**)

- Equivalently, for each player i:

$$a_i^* \in BR(a_{-i}^*) = argmax_b\, u_i(b, a_{-i}^*)$$

- Compared to DSE (a DES is a NE, the other way is generally not true):

$$a_i^* = BR(a_{-i}), \forall\, a_{-i}$$

Finding (pure) Nash Equilibria by hand

- As player 1: For each column, find the best response, underscore it.

Player 2 Player 1	*L*	*R*
T	2, 1	0, 0
B	0, 0	1, 2

Finding (pure) Nash Equilibria by hand

- As player 2: For each row, find the best response, upper-score it.

Player 2 Player 1	*L*	*R*
T	2, 1	0, 0
B	0, 0	1, 2

Finding (pure) Nash Equilibria by hand

- Entries with both lower and upper bars are pure NEs.

Player 2 / Player 1	*L*	*R*
T	2, 1	0, 0
B	0, 0	1, 2

Pure Nash Equilibrium may not exist

So far, pure strategy: each player picks a deterministic strategy. But:

Player 2 Player 1	*rock*	paper	*scissors*
rock	0, 0	-1, 1	1, -1
paper	1, -1	0, 0	-1, 1
scissors	-1, 1	1, -1	0, 0

Mixed Strategies

Can also randomize actions: "**mixed**"

- Player i assigns probabilities x_i to each action

$$x_i(a_i), \text{ where } \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0$$

- Now consider **expected rewards**

$$u_i(x_i, x_{-i}) = E_{a_i \sim x_i, a_{-i} \sim x_{-i}} u_i(a_i, a_{-i})$$
$$= \sum_{a_i} \sum_{a_{-i}} x_i(a_i) x_{-i}(a_{-i}) u_i(a_i, a_{-i})$$

Mixed Strategy Nash Equilibrium

Consider the mixed strategy *x** = (*x*$_1$*, …, *x*$_n$*)

- This is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-1}^*) \geq u_i(x_i, x_{-i}^*) \quad \forall x_i \in \Delta_{A_i}, \forall i \in \{1, \ldots, n\}$$

Better than doing anything else, "**best response**"

Space of probability distributions

- Intuition: nobody can **increase expected reward** by changing only their own strategy.

Mixed Strategy Nash Equilibrium

Example: $x_1(.) = x_{2(.)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Player 2 Player 1	*rock*	paper	*scissors*
rock	0, 0	-1, 1	1, -1
paper	1, -1	0, 0	-1, 1
scissors	-1, 1	1, -1	0, 0

Finding Mixed NE in 2-Player Zero-Sum Game

Example: Two Finger Morra. Show 1 or 2 fingers. The "even player" wins the sum if the sum is even, and vice versa.

odd / even	*f1*	f2
f1	2, -2	-3, 3
f2	-3, 3	4, -4

Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Two Finger Morra. Two-player zero-sum game. No pure NE:

odd / even	*f1*	f2
f1	2, -2	-3, 3
f2	-3, 3	4, -4

Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Suppose odd's mixed strategy at NE is (q, 1-q), and even's (p, 1-p)
By definition, p is best response to q: $u_1(p,q) \geq u_1(p',q) \forall p'$.

But $u_1(p,q) = pu_1(f_1,q) + (1-p)u_1(f_2,q)$
Average is no greater than components
➔ $u_1(p,q) = u_1(f_1,q) = u_1(f_2,q)$

	odd / even	*f1* (q)	f2 (1-q)
p	*f1*	2, -2	-3, 3
1-p	f2	-3, 3	4, -4

Finding Mixed NE in 2-Player 2-action Zero-Sum Game

$$u_1(f_1, q) = u_1(f_2, q)$$
$$2q + (-3)(1-q) = (-3)q + 4(1-q)$$
$$q = \frac{7}{12}$$

Similarly, $u_2(p, f_1) = u_2(p, f_2)$

$$p = \frac{7}{12}$$

At this NE, even gets -1/12, odd gets 1/12.

		q	1-q
	odd / even	*f1*	f2
p	*f1*	2, -2	-3, 3
1-p	f2	-3, 3	4, -4

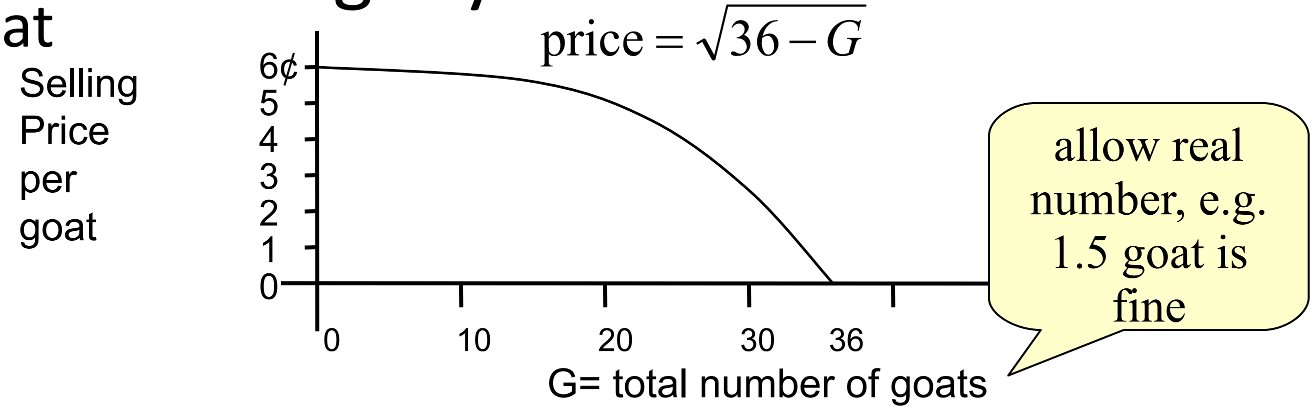
Properties of Nash Equilibrium

Major result: (Nash ’51)

- Every finite (players, actions) game has at least one Nash equilibrium
  - But not necessarily **pure** (i.e., deterministic strategy)
- Could be more than one
- Searching for Nash equilibria: computationally **hard**.
  - Exception: two-player zero-sum games (linear program).

Pure NE in an Infinite game: The tragedy of the Commons

- Price per goat



- How many goats should one (out of n) rational farmer graze?

- How much would the farmer earn?

Continuous Action Game

- Each farmer has infinite number of strategies $g_i \in [0,36]$
- The value for farmer *i*, when the *n* farmers play at $(g_1, g_2, \ldots, g_n)$ is

$$u_i(g_1, g_2, \ldots, g_n) = g_i \sqrt{36 - \sum_{j \in [n]} g_j}$$

- Assume a pure Nash equilibrium exists.
- Assume (by apparent symmetry) the NE is $(g^*, g^*, \ldots, g^*)$.

Finding g*

- $u_i(g_1, g_2, \dots, g_n) = g_i\sqrt{36 - \sum_j g_j}$
- g* is the best response to others (g*,…, g*)

$$g^* = argmax_{h\in[0,36]} u_i(g^*, \dots, h, \dots, g^*)$$

i-th argument

$$= argmax_h h\sqrt{36 - (n-1)g^* - h}$$

Finding g*

$$g^* = argmax_h h\sqrt{36 - (n-1)g^* - h}$$

- Taking derivative w.r.t. h of the RHS, setting to 0:

$$g^* = \frac{72 - 2(n-1)g^*}{3}$$

$$g^* = \frac{72}{2n+1}$$

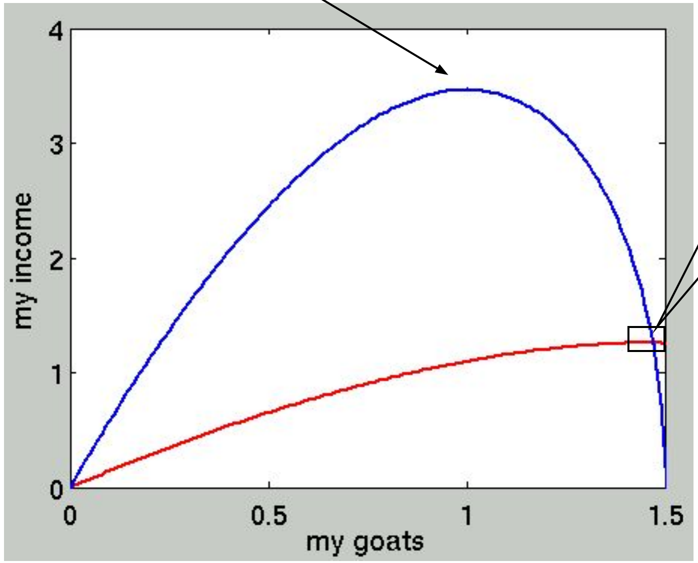
So what?

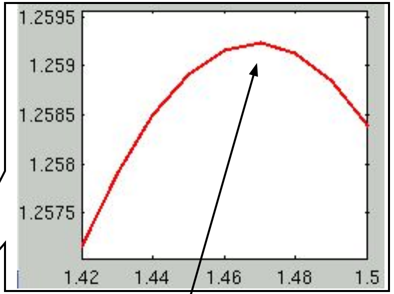
The tragedy of the Commons

- Say there are n=24 farmers. Each would rationally graze $g_i^* = 72/(2*24+1) = 1.47$ goats
- Each would get $g_i\sqrt{36-\sum_{j=1}^{n} g_j}$ = 1.25¢
- But if they cooperate and each graze only 1 goat, each would get 3.46¢

The tragedy of the Commons

If all 24 farmers agree on the same number of goals to raise, 1 goat per farmer would be optimal



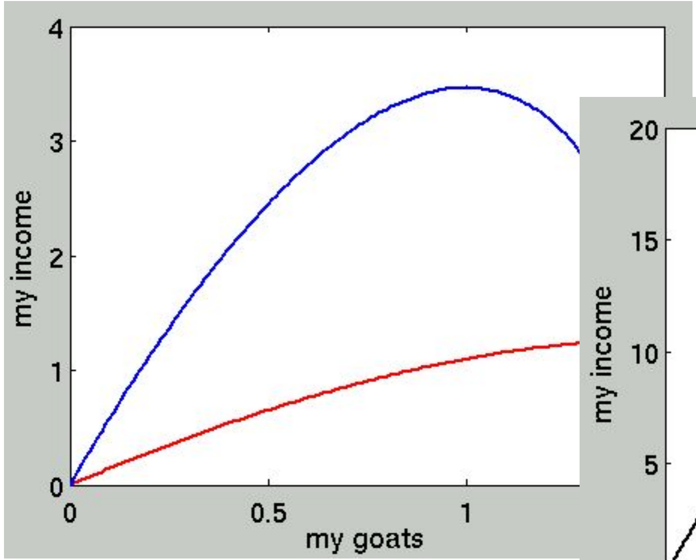


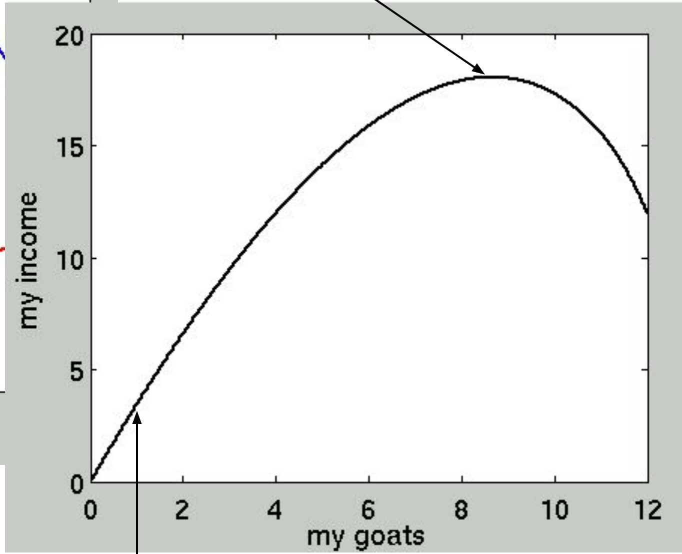
If the other 23 farmers play the N.E. of 1.47 goats each, 1.47 goats would be optimal

The tragedy of the Commons

If all 24 farmers agree on the same number of goals to raise, 1 goat per farmer would be optimal

But this is not a N.E.! A farmer can benefit from cheating (other 23 play at 1):





‘by rule’

The tragedy

- Rational behaviors lead to sub-optimal solutions!
- Maximizing individual welfare not necessarily maximizes social welfare
- What went wrong?

Shouldn't have allowed free grazing?

It's not just the  : the use of the atmosphere and the oceans for dumping of pollutants.

Mechanism design: designing the rules of a game

Break & Quiz

Q 2.1: Which of the following is true

(i) Rock/paper/scissors has a dominant pure strategy

(ii) There is no Nash equilibrium for rock/paper/scissors

- A. Neither
- B. (i) but not (ii)
- C. (ii) but not (i)
- D. Both

Break & Quiz

Q 2.1: Which of the following is **false**?

(i) Rock/paper/scissors has a dominant pure strategy

(ii) There is no Nash equilibrium for rock/paper/scissors

- A. Neither
- **B. (i) but not (ii)**
- C. (ii) but not (i)
- D. Both

Break & Quiz

Q 2.1: Which of the following is **false**?

(i) Rock/paper/scissors has a dominant pure strategy

(ii) There is no Nash equilibrium for rock/paper/scissors

- A. Neither (There is a mixed strategy Nash equilibrium)
- **B. (i) but not (ii)**
- C. (ii) but not (i) (i is indeed false: easy to check that there's no deterministic dominant strategy)
- D. Both (Same as A)

Break & Quiz

Q 2.2: Which of the following is true

(i) Nash equilibria require each player to know other players' strategies

(ii) Nash equilibria require rational play

- A. Neither
- B. (i) but not (ii)
- C. (ii) but not (i)
- D. Both

Break & Quiz

Q 2.2: Which of the following is **true**

(i) Nash equilibria require each player to know other players' strategies

(ii) Nash equilibria require rational play

- A. Neither
- B. (i) but not (ii)
- C. (ii) but not (i)
- **D. Both**

Break & Quiz

Q 2.2: Which of the following is true

(i) Nash equilibria require each player to know other players' strategies

(ii) Nash equilibria require rational play

- A. Neither (See below)
- B. (i) but not (ii) (Rational play required: i.e., what if prisoners desire longer jail sentences?)
- C. (ii) but not (i) (The basic assumption of Nash equilibria is knowing all of the strategies involved)
- D. **Both**

Summary

- Intro to game theory
  - Characterize games by various properties
- Mathematical formulation for simultaneous games
  - Normal form, dominance, Nash equilibria, mixed vs pure