CS 540 Introduction to Artificial Intelligence

Statistics & Math Review

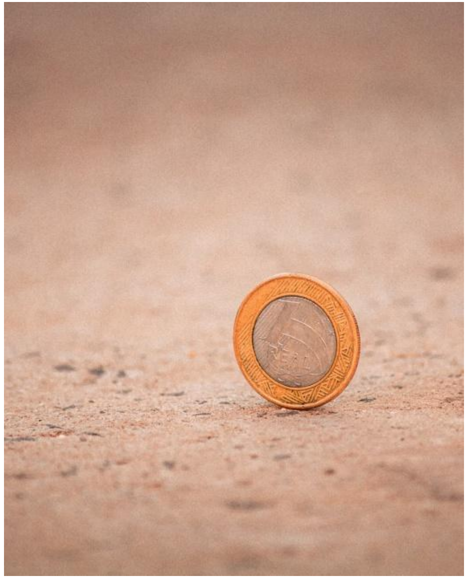
University of Wisconsin-Madison

Spring 2022

Samples and Estimation

- Usually, we don't know the distribution P
  - Instead, we see a bunch of samples

- Typical statistics problem: **estimate parameters** from samples
  - Estimate probability *P(H)*
  - Estimate the mean $E[X]$
  - Estimate parameters $P_\theta(X)$



Samples and Estimation

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  - Estimate the mean $E[X]$
  - Estimate parameters $P_\theta(X)$

- Example: Bernoulli with parameter *p*
  - Mean $E[X]$ is p

Examples: Sample Mean

- Bernoulli with parameter *p*
- See samples $x_1, x_2, \ldots, x_n$
  - Estimate mean with **sample mean**

$$\hat{\mathbb{E}}[X] = \frac{1}{n}\sum_{i=1}^{n} x_i$$

  - No different from counting heads

Break & Quiz

Q 2.1: You see samples of *X* given by [0,1,1,2,2,0,1,2]. Empirically estimate $E[X^2]$

A. 9/8
B. 15/8
C. 1.5
D. There aren't enough samples to estimate $E[X^2]$

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Q 2.2: You are empirically estimating *P(X)* for some random variable *X* that takes on 100 values. You see 50 samples. How many of your *P(X=a)* estimates might be 0?

A. None.
B. Between 5 and 50, exclusive.
C. Between 50 and 100, inclusive.
D. Between 50 and 99, inclusive.

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D. **Between 50 and 99, inclusive.**

Estimating Multinomial Parameters

- k-sized die (special case: k=2 coin)
- Face i has probability p_i, for $i=1..k$
- In n rolls, we observe face i showing up n_i times

$$\sum_{i=1}^{k} n_i = n$$

- Estimate $(p_{1,\ldots,} p_k)$ from this data $(n_{1,\ldots,} n_k)$

Maximum Likelihood Estimate (MLE)

- The MLE of multinomial parameters $(\widehat{p_1}, \dots, \widehat{p_k})$

$$\widehat{p_i} = \frac{n_i}{n}$$

- "frequency estimate"

Regularized Estimate

- Equivalent to a specific Maximum A Posterori (MAP) estimate, or smoothing
- Hyperparameter $\epsilon > 0$

$$\widehat{p_i} = \frac{n_i + \epsilon}{n + k\epsilon}$$

- Avoids zero when *n* is small
- Biased, but has smaller variance

Estimating 1D Gaussian Parameters

- Gaussian distribution $N(\mu, \sigma^2)$
- Observe *n* data points from this distribution

$$x_1, \dots, x_n$$

- Estimate μ, σ^2 from this data

Estimating 1D Gaussian Parameters

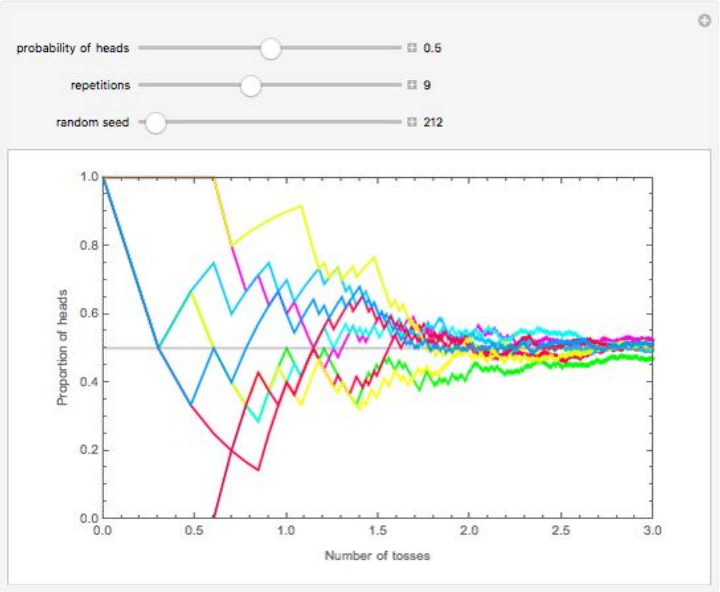
- Mean estimate $\hat{\mu} = \frac{x_1 + \cdots + x_n}{n}$
- Variance estimates
  - Unbiased $s^2 = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})^2}{n-1}$
  - MLE $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})^2}{n}$

Estimation Theory

- How do we know that the sample mean is a good estimate of the true mean?
  - Concentration inequalities

$$P(|\mathbb{E}[X] - \hat{\mathbb{E}}[X]| \geq t) \leq \exp(-2nt^2)$$

  - Law of large numbers
  - Central limit theorems, etc.



Wolfram Demo