CS 540 Introduction to Artificial Intelligence

Logic

University of Wisconsin-Madison

Spring, 2022

Logic & AI

Why are we studying logic?

- **Traditional** approach to AI ('50s-'80s)
  - "Symbolic AI"
  - The Logic Theorist – 1956
    - Proved a bunch of theorems!
- Logic also the language of:
  - Knowledge rep., databases, etc.

Symbolic Techniques in AI

Lots of systems based on symbolic approach:

- Ex: expert systems, planning, more
- Playing great chess

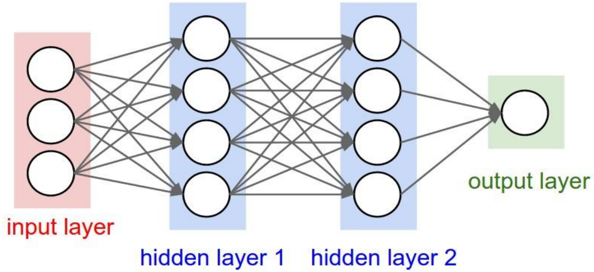
- **Less popular recently**!

J. Gardner

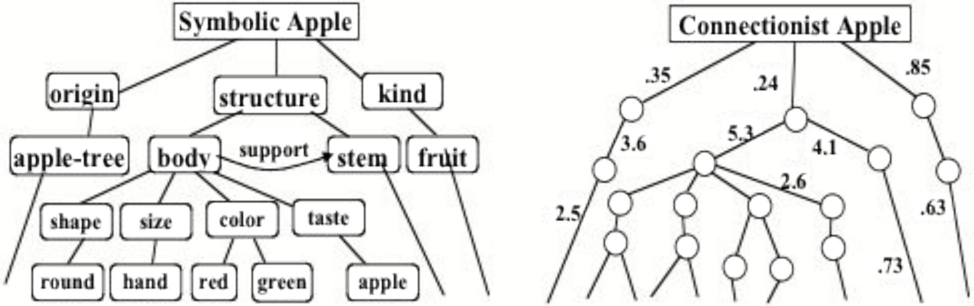
Symbolic vs Connectionist

Rival approach: **connectionist**

- Probabilistic models
- Neural networks
- **Extremely popular** last 20 years



Stanford CS231n

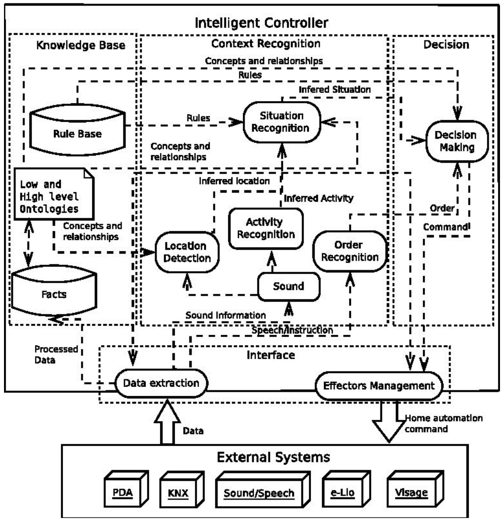


M. Minsky

Symbolic vs Connectionist

Analogy: Logic versus probability

- Which is better?
- Future: combination; best-of-both-worlds
  - Actually been worked on:
  - **Example**: Markov Logic Networks

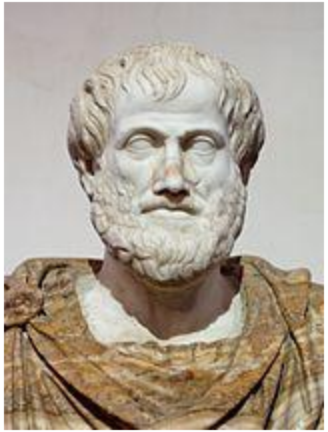


Outline

- Introduction to logic
  - Arguments, validity, soundness
- Propositional logic
  - Sentences, semantics, inference
- First order logic (FOL)
  - Predicates, objects, formulas, quantifiers

Basic Logic

- Arguments, premises, conclusions
  - Argument: a set of sentences (premises) + a sentence (a conclusion)
  - **Validity:** argument is valid iff it’s necessary that if all premises are true, the conclusion is true
  - **Soundness:** argument is sound iff valid & premises true
  - **Entailment:** when valid arg., premises entail conclusion



Propositional Logic Basics

Logic Vocabulary:

- Sentences, symbols, connectives, parentheses
  - Symbols: P, Q, R, … (**atomic** sentences)
  - Connectives:

$\wedge$	and	[conjunction]
$\vee$	or	[disjunction]
$\Rightarrow$	implies	[implication]
$\Leftrightarrow$	is equivalent	[biconditional]
$\neg$	not	[negation]

  - Literal: P or negation ¬P

Propositional Logic Basics

Examples:

- $(P \lor Q) \Rightarrow S$
  - “If it is cold or it is raining, then I need a jacket”
- $Q \Rightarrow P$
  - “If it is raining, then it is cold”
- $\neg R$
  - “It is not hot”

Propositional Logic Basics

Several rules in place

- Precedence: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- Use parentheses when needed
- Sentences: **well-formed** or not well-formed:

$P \Rightarrow Q \Rightarrow S$ **not well-formed (not associative!)**

Sentences & Semantics

- Think of symbols as defined by user
- Sentences: built up from symbols with connectives

  - **Interpretation**: assigning True / False to symbols
  - **Semantics**: interpretations for which sentence evaluates to True
  - **Model**: (of a set of sentences) interpretation for which all sentences are True

Evaluating a Sentence

- Example:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
$false$	$false$	$true$	$false$	$false$	$true$	$true$
$false$	$true$	$true$	$false$	$true$	$true$	$false$
$true$	$false$	$false$	$false$	$true$	$false$	$false$
$true$	$true$	$false$	$true$	$true$	$true$	$true$

- Note:
  - If P is false, P $\Rightarrow$ Q is true regardless of Q ("5 is even implies 6 is odd" is True!)
  - Causality unneeded: ("5 is odd implies the Sun is a star" is True!)

Evaluating a Sentence: Truth Table

- **Ex**:

P	Q	R	¬P	Q∧R	¬P∨Q∧R	¬P∨Q∧R⇒Q
0	0	0	1	0	1	0
0	0	1	1	0	1	0
0	1	0	1	0	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	0	0	1
1	1	0	0	0	0	1
1	1	1	0	1	1	1

- **Satisfiable**
  - There exists some interpretation where sentence true

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg p \rightarrow \neg q) \wedge r$

(ii) $(\neg p \vee \neg q) \rightarrow (p \vee \neg r)$

- A. Both
- B. Neither
- C. Just (i)
- D. Just (ii)

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg p \rightarrow \neg q) \land r$

(ii) $(\neg p \lor \neg q) \rightarrow (p \lor \neg r)$

- A. Both
- B. Neither
- **C. Just (i)**
- D. Just (ii)

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- a. $A \vee (\neg A \rightarrow B)$
- b. $A \vee B$
- c. $A \vee (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.2: Let A ="Aldo is Italian" and B ="Bob is English". Formalize "Aldo is Italian or if Aldo isn't Italian then Bob is English".

- **a. $A \vee (\neg A \rightarrow B)$**
- **b. $A \vee B$ (equivalent!)**
- c. $A \vee (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.3: How many different assignments can there be to

$$(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$$

- A. 2
- B. 2^n
- C. 2^{2n}
- D. 2n

Break & Quiz

Q 1.3: How many different assignments can there be to

$$(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \dots \vee (x_n \wedge y_n)$$

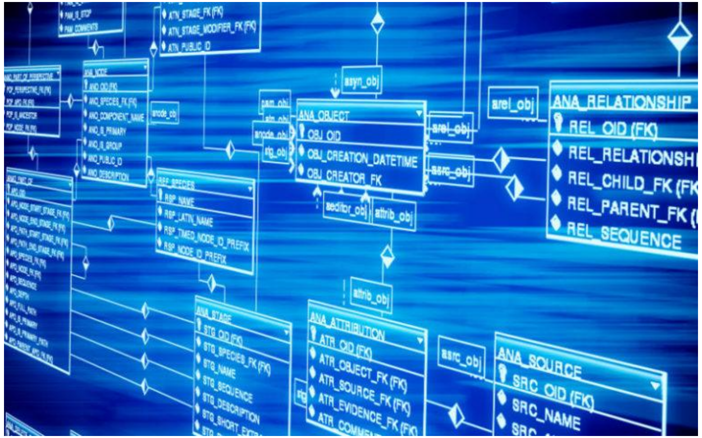
- A. 2
- B. 2^n
- **C. 2^{2n}**
- D. 2n

Knowledge Bases

- **Knowledge Base (KB)**: A set of sentences
  - Like a long sentence, connect with conjunction

Model of a KB: interpretations where all sentences are True

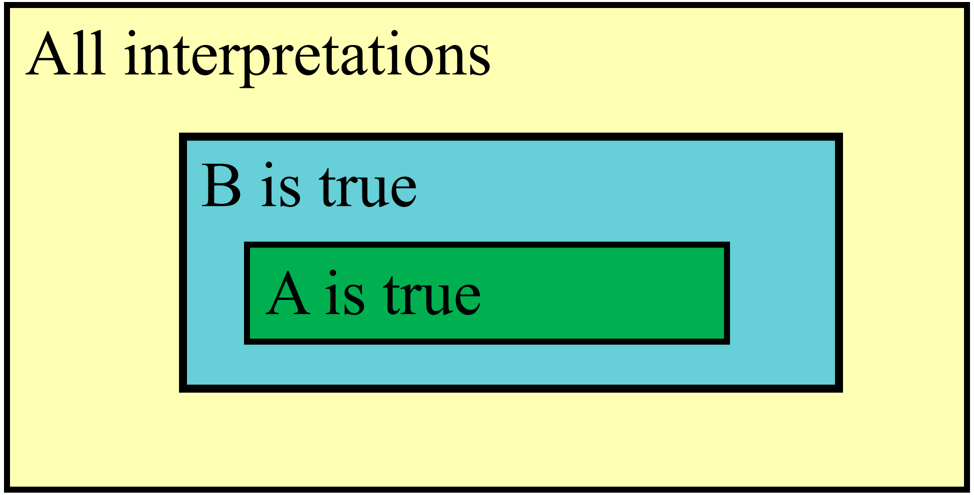
Goal: inference to discover new sentences



Entailment

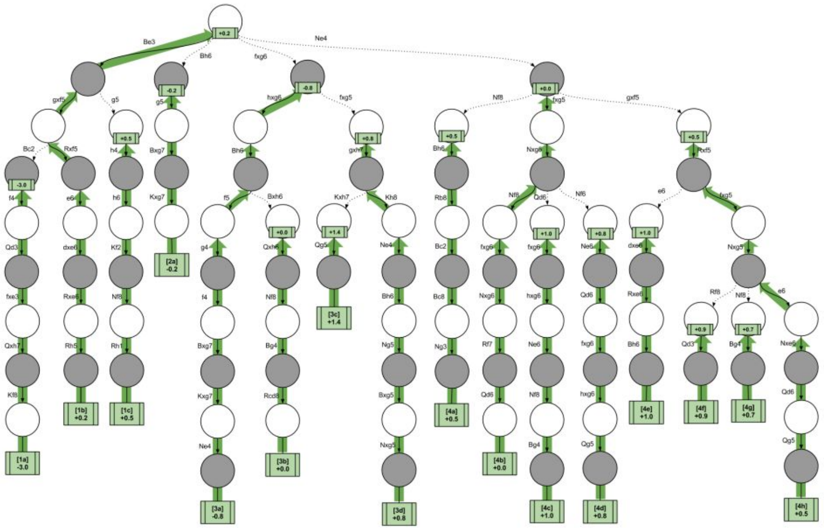
Entailment: a sentence logically follows from others

- Like from a KB. Write A ⊨ B
- A ⊨ B iff in every interpretation where A is true, B is also true



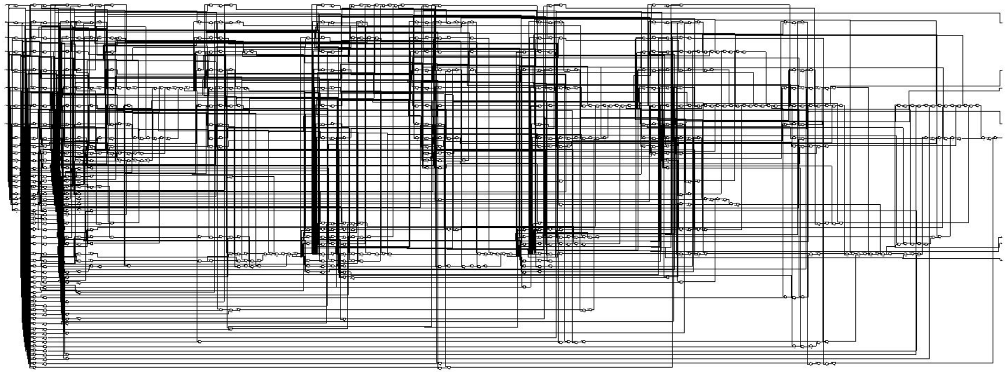
Inference

- Given a set of sentences (a KB), **logical inference** creates new sentences
  - Compare to prob. inference!
- **Challenges**:
  - Soundness
  - Completeness
  - Efficiency



Methods of Inference: **1. Enumeration**

- Enumerate all interpretations; look at the truth table
  - "Model checking"
- Downside: 2^n interpretations for n symbols



S. Leadley

Methods of Inference: **2. Using Rules**

- *Modus Ponens*: $(A \Rightarrow B, A) \models B$
- And-elimination
- Logical equivalences

Logical equivalences

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) && \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) && \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) && \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) && \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha && \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) && \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) && \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) && \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) && \text{de Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) && \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) && \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) && \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

You can use these equivalences to modify sentences.

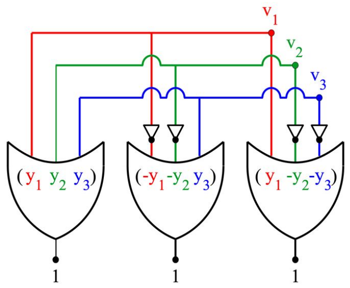
Methods of Inference: **3. Resolution**

- Convert to special form and use a single rule
- **Conjunctive Normal Form** (CNF)

$$(\neg \underbrace{A \vee B \vee C)}_{\text{a clause}} \wedge (\neg B \vee A) \wedge (\neg C \vee A)$$

Conjunction of clauses; each clause disjunction of literals

- Simple rules for converting to CNF



Conjunctive Normal Form (CNF)

$$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})$$

- Replace all $\Leftrightarrow$ using biconditional elimination
- Replace all $\Rightarrow$ using implication elimination
- Move all negations inward using
  -double-negation elimination
  -de Morgan's rule
- Apply distributivity of $\lor$ over $\land$

Convert example sentence into CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$ starting sentence

$(B_{1,1} \Rightarrow (P_{1,2} \lor P_{2,1})) \land ((P_{1,2} \lor P_{2,1}) \Rightarrow B_{1,1})$
biconditional elimination

$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg(P_{1,2} \lor P_{2,1}) \lor B_{1,1})$
implication elimination

$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land ((\neg P_{1,2} \land \neg P_{2,1}) \lor B_{1,1})$
move negations inward

$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})$
distribute $\lor$ over $\land$

Resolution steps

- Given KB and β (query)
- Add $\neg\beta$ to KB, show this leads to empty (False. Proof by contradiction)
- Everything needs to be in CNF
- Example KB:
  - $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
  - $\neg B_{1,1}$
- Example query: $\neg P_{1,2}$

Resolution preprocessing

- Add $\neg \beta$ to KB, convert to CNF:

  a1: $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1})$

  a2: $(\neg P_{1,2} \lor B_{1,1})$

  a3: $(\neg P_{2,1} \lor B_{1,1})$

  b: $\neg B_{1,1}$

  c: $P_{1,2}$

- Want to reach goal: *empty*

Resolution

- Take any two clauses where one contains some symbol, and the other contains its complement (negative)

$$P \vee Q \vee R \qquad\qquad \neg Q \vee S \vee T$$

- Merge (resolve) them, throw away the symbol and its complement

$$P \vee R \vee S \vee T$$

- If two clauses resolve and there's no symbol left, you have reached *empty* (False). KB $\models \beta$
- If no new clauses can be added, KB does not entail β

Resolution example

a1: $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1})$

a2: $(\neg P_{1,2} \vee B_{1,1})$

a3: $(\neg P_{2,1} \vee B_{1,1})$

b: $\neg B_{1,1}$

c: $P_{1,2}$

Resolution example

a1: $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1})$

a2: $(\neg P_{1,2} \vee B_{1,1})$

a3: $(\neg P_{2,1} \vee B_{1,1})$

b: $\neg B_{1,1}$

c: $P_{1,2}$

Step 1: resolve a2, c: $B_{1,1}$

Step 2: resolve above and b: *empty*

Efficiency of the resolution algorithm

- Run time can be exponential in the worst case
  - Often much faster
- Factoring: if a new clause contains duplicates of the same symbol, delete the duplicates

$$P \lor R \lor P \lor T \;\; □ \; P \lor R \lor T$$

- If a clause contains a symbol and its complement, the clause is a tautology and useless, it can be thrown away

a1: $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1})$
a2: $(\neg P_{1,2} \lor B_{1,1})$
□ $P_{1,2} \lor P_{2,1} \lor \neg P_{1,2}$ (tautology, throw away)

Break & Quiz

Q 2.1: What is the CNF for $(\neg p \land \neg(p \Rightarrow q))$

- A. $(\neg p \land \neg(p \Rightarrow q))$
- B. $(\neg p) \land (\neg p \lor \neg q)$
- C. $(\neg p \lor q) \land (p \lor \neg q) \land (p \lor q)$
- D. $(\neg p) \land (p) \land (\neg q)$

Break & Quiz

Q 2.1: What is the CNF for $(\neg p \land \neg(p \Rightarrow q))$

- A. $(\neg p \land \neg(p \Rightarrow q))$
- B. $(\neg p) \land (\neg p \lor \neg q)$
- C. $(\neg p \lor q) \land (p \lor \neg q) \land (p \lor q)$
- **D. $(\neg p) \land (p) \land (\neg q)$**

Break & Quiz

Q 2.2: Which has more rows: a truth table on *n* symbols, or a joint distribution table on *n* binary random variables?

- A. Truth table
- B. Distribution
- C. Same size
- D. It depends

Break & Quiz

Q 2.2: Which has more rows: a truth table on *n* symbols, or a joint distribution table on *n* binary random variables?

- A. Truth table
- B. Distribution
- **C. Same size**
- D. It depends

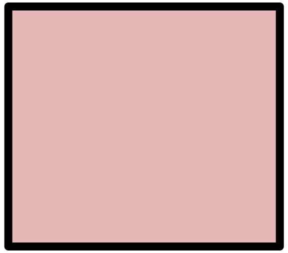
First Order Logic (FOL)

Propositional logic has some limitations

- Ex: how to say “all squares have four sides”
- No context, hard to generalize; express facts

FOL is a more expressive logic; works over

- Facts, Objects, Relations, Functions



First Order Logic syntax

- **Term**: an object in the world
  - **Constant**: Alice, 2, Madison, Green, …
  - **Variables**: x, y, a, b, c, …
  - **Function**($term_1$, …, $term_n$)
    - Sqrt(9), Distance(Madison, Chicago)
    - Maps one or more objects to another object
    - Can refer to an unnamed object: LeftLeg(John)
    - Represents a user defined functional relation

- A **ground term** is a term without variables.

FOL syntax

- **Atom**: smallest T/F expression
  - **Predicate**($term_1$, …, $term_n$)
    - Teacher(Jerry, you), Bigger(sqrt(2), x)
    - Convention: read "Jerry (is)Teacher(of) you"
    - Maps one or more objects to a truth value
    - Represents a user defined relation
  - **$term_1$ = $term_2$**
    - Radius(Earth)=6400km, 1=2
    - Represents the equality relation when two terms refer to the same object

FOL syntax

- **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: $\wedge \vee \neg \Rightarrow \Leftrightarrow$
    - Less(x,22) $\wedge$ Less(y,33)
  - Complex sentence using quantifiers $\forall$, $\exists$
- Sentences are evaluated under an interpretation
  - Which objects are referred to by constant symbols
  - Which objects are referred to by function symbols
  - What subsets defines the predicates

FOL quantifiers

- Universal quantifier: **∀**
- Sentence is true **for all** values of x in the domain of variable x.

- Main connective typically is ⇒
  - Forms if-then rules
  - “all humans are mammals”

    $$\forall x \ \mathbf{human(x)} \Rightarrow \mathbf{mammal(x)}$$
  - Means if x is a human, then x is a mammal

FOL quantifiers

- Existential quantifier: $\exists$
- Sentence is true **for some** value of x in the domain of variable x.

- Main connective typically is $\wedge$
  - "some humans are male"

    $$\exists x \; \mathbf{human(x)} \wedge \mathbf{male(x)}$$

  - Means there is an x who is a human and is a male