CS 577: Introduction to Algorithms	Homework 5a
Out: 11/16/06	**Due: 11/28/06**

Ground Rules

- Grading. You will be graded on the correctness as well as clarity of your solutions. You are required to prove any claims that you make. In particular, when you are asked to design an algorithm, you must argue its correctness and running time.
- Collaboration. You are allowed to discuss questions with other people in the class. However, **you must solve and write your answers yourself without any help**. You must also give explicit citations to any sources besides the textbook and class notes, including discussions with classmates. Solutions taken from external sources such as the WWW, even if cited, will receive no credit unless there is significant "value added". In cases of doubt, you may be asked to explain your answer to the instructor and this will determine your grade.
- Lateness. Please see the class webpage for details on the lateness policy.
- This homework is due right after Thanksgiving. Start working early. Plan your work in such a way that you have the opportunity to put some problems on the back burner for a while and revisit them later. Good luck!

Problems

1. **(Nuts & Bolts.)** You are given n nuts and n bolts of different sizes. Each nut fits exactly one bolt. Your job is to pair up the nuts with their matching bolts. You are allowed to compare nuts to bolts by trying to fit one into the other. The result of such a comparison is that the nut fits exactly with the bolt, or the nut is bigger, or the bolt is bigger. You cannot compare nuts to nuts or bolts to bolts. Give a randomized algorithm for solving this problem that runs in time $O(n \log n)$ in expectation.

2. Problem 13.9 and 13.10 in the textbook (p. 788–789).

   Extra credit: For problem 13.9 give a strategy that accepts the highest bid with probability 1/e.

3. **(The Chernoff bound.)** When we toss a coin with a bias of p a total of n times, the expected number of heads we get is np, however, there is some small probability that the actual number of heads will be very small or very large. As n gets large, this probability gets negligibly small. In particular, a well-known theorem in probability theory, the *Chernoff bound*, states that for any number $\epsilon > 0$, the probability that the number of heads is smaller than $(1-\epsilon)np$ is at most $e^{-\frac{1}{2}np\epsilon^2}$. (A similar bound holds for the probability that the number of heads is much larger than np.)

   As an example, suppose that the expected number of heads np is 1000, then the probability that you see less than 990 heads (i.e. $\epsilon = 0.01$) is at most $1/e^{0.05} \approx 0.95$, but the probability that you see less than 900 heads ($\epsilon = 0.1$) is at most $1/e^5 \approx 0.006$.

   Use the Chernoff bound to solve problem 13.15 in the textbook (p. 791).

4. **(A numbers game.)** Consider the following game. A friend writes down two arbitrary numbers (+ve or -ve) on two slips of paper and then puts one in one hand and the other in the other hand. You then pick a hand uniformly at random and see the number in it. You then can either keep the number you saw or else return it and get the other number. Say you end up with the number a and the other number was b. Then, your gain is $a - b$.

   For a given (possibly randomized) strategy S, let $E[S|x, y]$ denote its expected gain, given that the two numbers that your friend has in his hands are x and y. (Note that the expectation is taken over the randomness in the strategy, as well as over the randomness in picking a hand – you see x with probability $1/2$ and y with probability $1/2$.)

(a) Consider the strategy S = "if the first number I see is ≥ -17, then I keep it, else I switch." What is $E[S|x, y]$ in terms of x and y? (You could get different answers depending on what x and y are; write all possible cases.)

(b) Give a randomized strategy S such that $E[S|x, y] > 0$ for all $x \neq y$.

HAPPY THANKSGIVING!!