CS787: Advanced Algorithms

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13.1 Routing to minimize congestion

We are given a graph $G = (V, E)$ and k "commodities". Each commodity i has a source s_i and a destination t_i. The goal is to find an (s_i, t_i) path in the graph G for every commodity i, while minimizing the maximum congestion along any edge. The latter can be written formally as
Minimize congestion $C = max_e|\{i : P_i \ni e\}|$, where P_i is the (s_i, t_i) path of commodity i and e is an edge.

It can be noted that this problem is similar to a network flow problem. But it is not a network flow problem because for each commodity i, we need to have a single path between s_i and t_i. We cannot have the flow splitting along multiple branches.

This problem is an NP-hard problem. So we will solve it by formulating it as an ILP problem, relaxing it to an LP problem, solving the LP, and rounding the solution to an ILP solution.

13.1.1 ILP formulation with exponential number of variables and constraints

Let $i = 1, \ldots, k$ be the k commodities.
Let P_i be the set of all paths from s_i to t_i.
We have a variable x_P for every $P \in P_i$, $\forall i$

Minimze t s.t

$$\sum_{P \in P_i} x_P = 1, \forall i \tag{13.1.1}$$

$$\sum_i \sum_{P \in P_i, e \in P} x_P \leq t, \forall e \in E \tag{13.1.2}$$

$$t \geq 0 \tag{13.1.3}$$

$$x_P \in \{0, 1\} \forall P \tag{13.1.4}$$

Note that the actual objective function which is in a min-max form is not linear. So, we employ a trick of introducing a new variable t. We introduce the constraint (13.1.2) that congestion on any edge $\leq$ t, and so all we need to do now is minimze t, which is equivalent to minimizing the maximum congestion. The constaint 13.1.1 makes sure that we select exactly one path for each commodity.

The problem with this ILP formulation is that we can have an exponential number of paths between s_i and t_i (for example, in a completely connected graph). So even if we relax it to an LP problem, we will still have an exponential number of variables and constraints. We would like a formulation with a polynomial (polynominal in $|V|$, $|E|$, and k) number of variables and constraints. Next we consider an alternative ILP formulation.

13.1.2 ILP formulation with polynomial number of variables and constraints

Rather than looking at all the paths, we look at an edge granularity and see if a commodity is routed along any edge.
We have variables $x_{e,i}$, $\forall e \in E, i = 1, \ldots, k$
$x_{e,i} = 1$, if $e \in P_i$, and 0 otherwise. [P_i is the chosen (s_i, t_i) path of commodity i]
The ILP formulation is

Minimize t s.t

$$\sum_{e \in \delta^+(t_i)} x_{e,i} = \sum_{e \in \delta^-(s_i)} x_{e,i} = 1 \forall i \tag{13.1.5}$$

$$\forall i, \forall v \neq s_i, t_i \sum_{e \in \delta^+(v)} x_{e,i} = \sum_{e \in \delta^-(v)} x_{e,i} \tag{13.1.6}$$

$$\sum_i x_{e,i} \leq t \forall e \in E \tag{13.1.7}$$

$$t \geq 0 \tag{13.1.8}$$

$$x_{e,i} \in \{0,1\} \tag{13.1.9}$$

In the above formulation, δ^+ indicates the flow coming into a vertex, and δ^- indicates the flow going out of a vertex. Constraint 13.1.5 makes sure that we have a unit flow being routed out of every s_i, and a unit flow being routed into every t_i. Constraint 13.1.6 is like flow conservation at every other vertex. Congestion along an edge is just the number of commodities being routed along that edge, and constraint 13.1.7 ensures that $t \geq$ congestion along any edge. The objective function is to minimze t.

The problem can be relaxed to an LP problem by letting $x_{e,i} \in [0, 1]$. It can be noted that this formulation (let's call it edge formulation) is equivalent to the formulation in Section 13.1.1 (let's call it path formulation). From the LP solution to this edge formulation, one can obtain the LP solution to the path formulation. (This is left as an exercise to the reader). Now, we have a fractional solution $\{x_P\}$ for each variable x_P. We also know that the solution to the LP $t \leq C^*$, where C^* is the optimal solution to the ILP. Now, we need to round the LP solution to an ILP solution such that we achieve a reasonable approximation to t, the LP solution.

13.1.3 Deterministic rounding

First, we can note that a deterministic rounding doesn't help us get a good approximation. For example, let us look at a deterministic strategy. A reasonable strategy would be to look at all $P \in P_i$, and round the one with highest value to 1, and others to 0. Say if P_i has n paths, it might be the case that in the LP solution, all the n paths have a weight of $\frac{1}{n}$. So while rounding, the value of a path can increase by a factor of n. Now the optimal congestion t on an edge is caused by some paths, and if the value of each of these paths increase by a factor of n during the rounding, we can only get an n factor approximation. Since n can be an exponential number, this is not a very useful approximation factor. Likewise it can be seen that other deterministic rounding strategies do not help in getting a good approximation factor.

13.1.4 Randomized rounding

For every commodity i, consider the probability distribution on P_i given by $\{x_P\}_{P \in P_i}$, and pick one of the P_i's according to this probability distribution. For example, say we have three paths with values 0.5, 0.3, and 0.2. Get a random number between 0 and 1. If the number is between 0 and 0.5, pick the first path. If the number is between 0.5 and 0.8, pick the second path, and if the number is between 0.8 and 1, pick the third path.

For any edge e, $\mathbf{Pr}[\text{commodity } i \text{ is routed along } e] = x_{e,i}$.
Let X_e be a random variable , and let X_e = number of commodities i with $P_i \ni e$. In other words, X_e is a random variable which indicates the level of congestion along an edge e. To get the $\mathbf{E}[X_e]$, we can use indicator random varibales.
Let $X_{e,i} = 1$, if $P_i \ni e$, and 0 otherwise.
$X_e = \sum_i X_{e,i}$
$\Rightarrow \mathbf{E}[X_e] = \sum_i \mathbf{E}[X_{e,i}]$ [By linearity of expectation]
$= \sum_i x_{e,i} \leq t$
This means that the expected value of congestion along any edge $\leq t$, the solution to the LP problem. Now if we can show that for any edge, $\mathbf{Pr}[X_e \geq \lambda t] \leq \frac{1}{n^3}$, the by the union bound, $\mathbf{Pr}[\exists$ edge e s.t $X_e \geq \lambda t] \leq \frac{|E|}{n^3} \leq \frac{1}{n}$, and we can get a λ approximation with a high probability.

$$X_e = \sum_i X_{e,i}$$

$$\mathbf{E}[X_e] = \mu \leq t$$

$$\mathbf{Pr}[X_e > \lambda t] \leq \mathbf{Pr}[X_e > \lambda \mu]$$

Using Chernoff's bounds:

$$\mathbf{Pr}[X_e > \lambda t] \leq \left(\frac{e^{\lambda-1}}{\lambda^\lambda}\right)^\mu$$

$$\approx \lambda^{-\lambda\mu}$$

$$\approx \lambda^{-\lambda}$$

(Taking $\mu \approx 1$, atleast edge picked once in expectation.) Now,

$$\lambda^\lambda = n^3 \Rightarrow \lambda = O\left(\frac{logn}{loglogn}\right)$$

. We have with high probability, λ approximation.

13.2 LP Duality

The motivation behind using an LP dual is they provide lower bounds on LP solutions. For instance, consider the following LP problem.
Minimize $7x + 3y$, such that

$$x + y \geq 2$$

$$3x + y \geq 4$$

$$x, y \geq 0$$

Say, by inspection, we get a solution $x = 1, y = 1$, with an objective function value of 10. We can show that this is the optimal solution in the following way.
Proof: If we multiply the constraints with some values and add them such that the coeffiecients of x and y are $<$ 7 and 3, respectively, we can get a lower bound on the solution. For instance, if we add the two constraints, we get $4x + 2y \geq 6$, and since $x, y \geq 0$, we have $7x + 3y > 4x + 2y \geq 6$. So, 6 is a lower bound to the optimal solution. Likewise, if we multiply the first constraint with 1, and the second constraint with 2, we have

$$7x + 3y = (x + y) + 2(3x + y) \geq 2 + 2(4) = 10$$

Hence, 10 is a lower bound on the solution, and so $(x = 1, y = 1)$ with an objective function value of 10 is an optimal solution. ■
Similar to the above example, LP duality is a general way of obtaining lower bounds on the LP solution. The idea is we multiply each constraint with a multiplier and add them, such that the sum of coefficients of any variable is $\leq$ the coefficient of the variable in the objective function. This gives us a lower bound on the LP solution. We want to choose the multipliers such that the lower bound is maximized (giving us the tightest possible lower bound).

13.2.1 Converting a primal problem to the dual form

To convert the primimal to the dual, we do the following.
1. For each constraint in the primal, we have a corresponding variable in the dual. (this variable is like the multiplier).
2. For each variable in the primal, we have a corresponding constraint in the dual. These constraints say that when we multiply the primal constraints with the dual variables and add them, the sum of coefficients of any primal variable should be less than or equal to the coefficient of the variable in the primal objective function.
3. The dual objective function is to maximize the sum of products of right hand sides of primal constraints and the corresponding dual variables. (This is maximizing the lower bound on the primal LP solutions).
The following is a more concrete example showing the primal to dual conversion.
A linear programming problem is of the form:

Minimize $\sum_i c_i x_i$, such that,

$$\sum_i A_{ij} x_i \geq b_j \forall j$$

$$x_i \geq 0 \forall i$$

We call this the primal LP. This LP can be expressed in the matrix form as:

Minimize $c^T x$, such that,

$$Ax \geq b$$

$$x \geq 0$$

The corresponding dual problem is: Maximize $\sum_j b_j y_j$, such that,

$$\sum_j A_{ij} y_j \leq c_i \forall i$$

$$y_j \geq 0 \forall j$$

Expressed in matrix form, the dual problem is, Maximize $b^T y$, such that

$$A^T y \leq c$$

$$y \geq 0$$

Note that the dual of a dual LP is the original primal LP.

Theorem 13.2.1 Weak LP duality theorem *If x is any primal feasible solution and y is any dual feasible solution, then $Val_P(x) \geq Val_D(y)$*

Proof:

$$\sum_i c_i x_i \geq \sum_i \left(\sum_j A_{ij} y_j \right) x_i$$

$$= \sum_j \left(\sum_i A_{ij} x_i \right) y_j$$

$$\geq \sum_j b_j y_j$$

■

So, the weak duality thoerem says that any dual feasible solution is a lower bound to the primal optimal solution. This is a particularly nice result in the context of approximation algorithms. In the previous lectures, we were solving the primal LP exactly and using the LP solution as a lower bound to the optimal ILP solution. By using the weak duality theorem, instead of solving the LP exactly to obtain a lower bound on the optimal value of a problem, we can (more easily) use any dual feasible solution to obtain a lower bound.

Theorem 13.2.2 Strong LP duality theorem *If the primal has an optimal solution x^* and the dual has an optimal solution y^*, then $c^T x^* = b^T y^*$, i.e, the primal and the dual have the same optimal objective function value.*

In general, if the primal is infeasible (there is no feasible point which satisfies all the constraints), the dual is unbounded (the optimal objective function value is unbounded). Similarly, if the dual is infeasible, the primal is unbounded. However, if both the primal and dual are feasible (have atleast one feasible point), the strong LP duality theorem says that the optimal solutions to the primal and the dual have the exact same objective function value.