CS 787: Advanced Algorithms	**Take-home final exam**
Out: 05/06/13	**Due: 05/13/13**

Guidelines

- This final is to be done individually. No collaboration is allowed. You are allowed to consult only the course textbook and lecture notes.
- Please attempt all questions. The point distribution is indicated with each problem.

Problems

1. **(3 points)** We are given an urn with one black ball and one white ball. We run the following process until the urn has exactly n balls: we pick a ball uniformly at random from the urn and return it along with another ball of the same color. Derive the distribution of the number of white balls in the urn at the end of the process.

2. **(6 points)** Let X be a random walk on a cycle of length n with self loops. In particular, at every step, X moves clockwise with probability $1/3$, counterclockwise with probability $1/3$, and stays put with probability $1/3$. Determine the mixing time of X.

   Hint: Couple X with another random walk Y in such a way that at every step the distance between them decreases with some probability. Analyze the coupling by thinking of the distance between X and Y as another random walk.

3. **(6 points)** Let Π_n denote the set of all permutations on the first n integers $\{1, 2, \cdots, n\}$. For a permutation $\pi \in \Pi_n$, let

   $$I(\pi) = |\{(i, j) : 1 \leq i < j \leq n \text{ and } \pi_i > \pi_j\}|$$

   be the number of inversions in π. Let X denote the number of inversions in a permutation drawn uniformly at random from the set Π_n.

   (a) Determine $\mathrm{E}[X]$.

   (b) Given an upper bound on the probability that X is more than $(1 + \epsilon)\mathrm{E}[X]$ for some $\epsilon > 0$.