| CS880: Topics in Algorithmic Game Theory | Homework 1 |
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| **Out: 2/8/2011** | **Due: 2/22/2011** |

## Guidelines

- You may discuss this homework with your classmates, but you must write the solutions yourself and submit them individually.
- You may consult any reading material on the class webpage for help with the homework, but *not* other research papers or websites.
- Partial progress on problems will get partial credit (as long as there are no serious mistakes).

## Exercises

1. (5 points) Consider a general mechanism design problem with a set $\Omega$ of outcomes and $n$ agents, where each agent $i$ has a private real-valued valuation $v_i(\omega)$ for each $\omega \in \Omega$. Suppose the function $f : \Omega \to \mathbb{R}$ has the form

   $$f(\omega) = c(\omega) + \sum_{i=1}^{n} w_i v_i(\omega),$$

   where $c$ is a publicly known function of the outcome, and where each $w_i$ is a nonnegative, public, agent-specific weight. Such a function is called an *affine maximizer*.

   Show that for every affine maximizer objective function $f$, there is a truthful mechanism that optimizes $f$ over $\Omega$.

   [**Hint:** Modify the VCG mechanism. Don't worry about individual rationality or computational efficiency.]

2. (5 points) In class we characterized the outcomes and payments for BIC in single-dimensional agent settings. This characterization explains what happens when agents behave strategically.

   Suppose instead of strategic interaction, we care about fairness. Consider a valuation profile, $\mathbf{v} = (v_1, \ldots, v_n)$, an allocation vector $\mathbf{x} = (x_1, \ldots, x_n)$, and a payment vector $\mathbf{p} = (p_1, \ldots, p_n)$. Here $x_i$ is the probability that $i$ is served, and $p_i$ is the expected payment of $i$ regardless of whether $i$ is served or not.

   Allocation $\mathbf{x}$ and payments $\mathbf{p}$ are *envy-free* for valuation profile $\mathbf{v}$ if no agent wants to unilaterally swap allocation and payment with another agent, i.e. if for all $i$ and $j$, $v_i x_i - p_i \geq v_i x_j - p_j$.

   Characterize envy-free allocations and payments (and prove your characterization correct). Unlike the BIC characterization, your characterization of payments will not be unique. Instead, characterize the minimum payments that are envy-free.

   [**Hint:** You should end up with a very similar characterization to that of BIC.]

3. Consider a simplification of the Google AdWords setting: There are $m$ advertisement slots that appear alongside search results and $n$ advertisers. Advertiser $i$ has value $v_i$ for a click. Slot $j$ has *click-through rate* $w_j$, meaning that if an advertiser is assigned slot $j$, then the advertiser will receive a click with probability $w_j$. Assume that the slots are ordered from highest click-through rate to lowest, i.e. $w_1 \geq w_2 \geq \cdots \geq w_m$.

   (a) (5 points) Find the envy-free outcome and payments with the maximum social welfare. Give a description and formula for the envy-free outcome and payments for each agent (feel free to specify your payment formula with a comprehensive picture).

   (b) (2 points) In the real AdWords problem, advertisers only pay if they receive a click, whereas the payments $\mathbf{p}$ we calculated are in expectation over all outcomes, both with and without the advertiser being clicked on. If we are going to charge advertisers only if they are clicked on, give a formula for calculating these payments $\mathbf{p}'$ from $\mathbf{p}$.

   (c) (5 points) The real AdWords problem is solved by auction. Design an auction that maximizes the social welfare in dominant strategy equilibrium. Give a formula for the payment rule of your auction (again, a comprehensive picture is fine). Compare your DSE payment rule to the envy-free payment rule. Draw some informal conclusions.

4. When designing revenue-optimal mechanisms in a Bayesian setting, "sufficient competition" can obviate the need for a reserve price. This problem demonstrates one way of making this statement precise.

   (a) (3 points) Consider a distribution $F$ that is regular in the sense that, if we let $\phi$ denote the corresponding virtual valuation function, then $\phi$ is a nondecreasing function of $v$. Prove that the expected virtual value $\phi(v)$ of a valuation $v$ drawn from $F$ is precisely zero.

   (b) (5 points) Consider selling $k \geq 1$ identical items to agents with valuations drawn i. i. d. from $F$. Prove that for every $n \geq k$, the expected revenue of the Vickrey auction (with no reserve) with $n + k$ agents is at least that of the Bayesian revenue-optimal auction for $F$ with $n$ bidders. (Thus, modest additional competition is at least as valuable as knowing the distribution $F$ and employing a corresponding optimal reserve price.)

   [**Hint:** Try the $k = 1$ case first for intuition and partial credit. In general, use Myerson's characterization of the expected revenue of a truthful auction and problem 4a above]

5. (0 points) The last homework problem is to begin thinking about and working on the course project. In particular, when turning in the homework you should include:

   - the name of your project partner (if you are planning to work with someone);
   - a short description of the topic for the project; and
   - an outline of the goals of the project.

   You should start thinking about this as soon as possible. Please see the course page:

   `http://pages.cs.wisc.edu/ shuchi/courses/880-S11/projects/projects.html`

   for a list of topic suggestions. Of course, you should also feel free to suggest other topics as well, and discuss possibilities with either of us.