CS880: Topics in Algorithmic Game Theory	Homework 2
Out: 2/22/2011	**Due: 3/8/2011**

Guidelines

Same as for HW1.

Problems

1. In a *procurement* auction, there is a single buyer and multiple sellers. Each seller i has an intrinsic and private value for their good (which we will call a *cost* c_i), and the buyer wants to obtain a desired subset of goods. In the simplest case, the buyer needs to acquire one good, and subject to this hard constraint, wants to pay as little as possible. The analog of the truthful VCG mechanism is to award the buyer the cheapest good and charge the buyer the second-lowest reported cost. In class we discussed a more general setting with a graph $G = (V, E)$ where each edge is a seller and the buyer's goal is to buy a path between two fixed end-points. This is called a path auction.
   (a) (2 points) Let P^* be the cheapest path between the given end-points and P' be the cheapest path between the end-points that is edge disjoint from P^* (that is, shares no edges with P^*). Assume that P' exists. Note that the path auction selects the path P^*. Show that in a path auction the buyer can end up paying up to $\Omega(n)$ times the cost of P' where n is the number of vertices in the graph.
   (b) (3 points) Next consider a setting where the buyer wants to buy a minimum spanning tree in the graph. Prove that the unique truthful payments in VCG (from the buyer to the edges) that always pay zero to unpicked edges are the following: for every edge e of the chosen MST T, let f denote the cheapest edge other than e that crosses the (unique) cut induced by the two connected components of $T - \{e\}$; then the payment from the buyer to edge e is c_f.
   (c) (3 points) For this part and the next, assume that edge costs in G are all distinct. Let T_1 be the MST and T_2 be the cheapest spanning tree that is edge disjoint from T_1. (Assume that such a tree exists.) Prove that there is a one-one mapping g from the edges of T_1 to the edges of T_2 such that for all $e \in T_1$, $T_1 - \{e\} \cup \{g(e)\}$ is a spanning tree of G.
   [Hint: You should assume and use Hall's Theorem, which states that a bipartite graph $H = (U, W)$ with $|U| = |W|$ has a perfect matching if and only if for every subset $S \subseteq U$ with neighbors $\Gamma(S) \subseteq W$ on the other side, $|S| \leq |\Gamma(S)|$. (The "only if" direction of this theorem is trivial; the "if" direction is not.) Consider a graph H with the edges of T_1 on one side and the edges of T_2 on the other side, connecting $e_1 \in T_1$ to $e_2 \in T_2$ iff $T_1 - \{e_1\} \cup \{e_2\}$ spans G.]
   (d) (3 points) Using parts (b) and (c), prove that the total payment made by the mechanism in (b) to the edges is at most the cost of the cheapest spanning tree that is edge-disjoint from the MST.

2. A commonly used assumption about distributions in economics and statistics is the *Monotone Hazard Rate* (MHR) condition. A distribution F is said to satisfy MHR if the *hazard rate* of the distribution, $h(z) = f(z)/(1 - F(z))$, is monotone non-decreasing in z. Many natural distributions such as the uniform, normal, and exponential, satisfy this condition.

(a) (1 points) Prove that a distribution satisfying MHR is regular (that is, it has a monotone non-decreasing virtual value function).

(b) (3 points) Let F be an MHR distribution with virtual value function ϕ. Prove that for all $z \geq \phi^{-1}(0)$, $z \leq \phi^{-1}(0) + \phi(z)$.

(c) (3 points) Consider a single parameter setting with an arbitrary feasibility constraint. Using part (b) prove that the revenue of Myerson's mechanism is at least half the total social welfare (i.e. the expected sum of values of the winners) that the mechanism achieves.

(d) (4 points) Consider a single agent single item setting. Prove that if the agent's value distribution satisfies MHR, then the probability that Myerson's mechanism sells to this agent is at least $1/e$.

[Hint: first (and for partial credit) prove the result for exponential distributions, that is, with $F(z) = 1 - e^{-z}$ for $z \geq 0$. What is the hazard rate of such a distribution? In general, try to relate the function $F(z)$ to the integral of the hazard rate $\int_0^z h(t)dt$.]

3. Consider the revenue maximization problem in the digital goods setting. Recall that in order to obtain a good approximation to the benchmark $\mathcal{G}_2$, we needed to achieve some consistency across the prices charged to different agents. The RSPE auction achieved this by splitting the set of agents into two parts and charging consistent prices within each of the two parts. In this problem we will develop a different mechanism to achieve consistency.

   For a vector b of bids and agent i, let $OPT(b) = \mathcal{G}_2(b)$, and $OPT_{-i}(b) = \mathcal{G}(b_{-i})$. Observe that for all i, $OPT(b)/2 \leq OPT_{-i}(b) \leq OPT(b)$.

   (a) (3 points) Let $r_1(x)$ and $r_2(x)$ denote the functions that round x (up or down) to the nearest odd power of 2 and the nearest even power of 2, respectively. (E.g., $r_1(12) = 2^3 = 8$ while $r_2(12) = 2^4 = 16$.) Prove that for every bid vector b, there is always a choice of $j \in \{1, 2\}$ such that $r_j(OPT(b)) \leq OPT(b)$ and also $r_j(OPT_{-i}(b)) = r_j(OPT(b))$ for every i.

   (b) (2 points) Consider the following randomized auction: (1) Choose $j \in \{1, 2\}$ uniformly at random; (2) for every i, let $R_i = r_j(OPT_{-i}(b))$, and consider hypothetically running the ProfitExtract subroutine on b; let S_i and p_i denote the winning set and price respectively; (3) offer price p_i to agent i.

   Prove that this auction is truthful.

   (c) (3 points) Prove that the expected revenue of the auction in part (b) is at least $OPT(b)/4$.

   [Hint: Let j be the index for which the function r_j achieves "consensus" across all i in the sense of part (a). Consider running ProfitExtract with a revenue target of $r_j(OPT(b))$.]

4. **(Extra credit)** Consider the revenue maximization problem in the digital goods setting with n agents and suppose that every agent's value is either 1 or h for some large known number h, with at least two values being h. Give a *deterministic* truthful mechanism that achieves a 2-approximation to $\mathcal{G}_2$.

   [Hint: In order to avoid the bad example that we saw in class, your mechanism must be "asymmetric", that is, sometimes charge different prices to agents i and j with identical vectors v_{-i} and v_{-j}.]