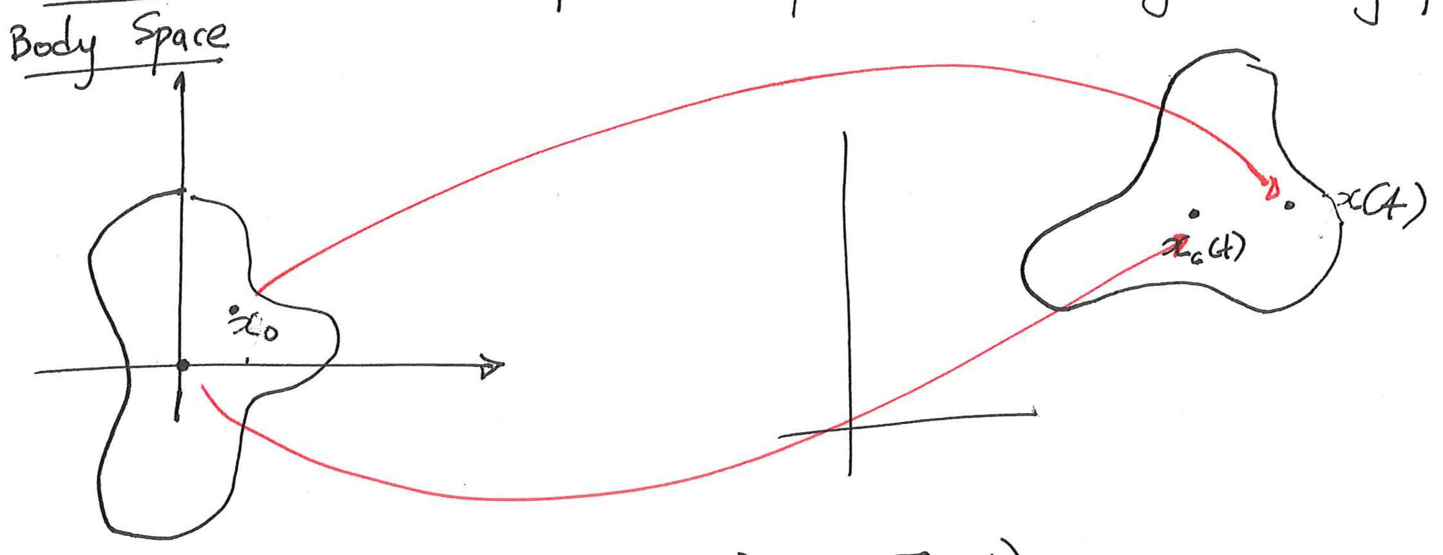
Rigid Body Dynamics

We previously discussed modeling, discretizing & simulating deformable bodies. Rigid bodies, on the other hand are a special case of solid models, and warrants specialized treatment

A rigid body may be positioned in various locations & orientations. To aid in our exposition, we define the "body space" to be a coordinate axis system which is fixed w.r.t. the rigid body. The relation between the body-space location of a point x_0 of the rigid body, and its world-space position is

Convention: The center of mass is placed at the origin in body-space.



$$\vec{x}(t) = R(t)\vec{x}_0 + \vec{x}_c(t)$$

Where : x_0 : Body-space position
$x(t)$: World-space position at time t
$R(t)$: Rotation matrix (same for all points)
$x_c(t)$: Position of the center of mass

What is the corresponding velocity of this point?

$$\vec{x}(t) = R(t)\vec{x}_0 + \vec{x}_c(t) \Rightarrow \vec{x}'(t) = R'(t)\vec{x}_0 + \vec{x}_c'(t) \Rightarrow$$

$$\Rightarrow \vec{v}(t) = R'(t)\vec{x}_0 + \vec{v}_c(t)$$

$v_c(t)$ = velocity of center of mass.

The question is... how do we express $R'(t)$?

Let's work it out : $R(t)$ is an orthogonal matrix, thus :

$$R(t)[R(t)]^T = I \Rightarrow \frac{d}{dt}\left\{R(t)[R(t)]^T\right\} = 0 \Rightarrow$$

$$\Rightarrow 0 = R'(t)[R(t)]^T + R(t)[R'(t)]^T \Rightarrow R'(t)[R(t)]^T = -\left\{R'(t)[R(t)]^T\right\}^T$$

This last expression, shows that $R'(t)\cdot[R(t)]^T$ is an anti-symmetric 3×3 matrix.

Theorem #1 If $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \in \mathbb{R}^3$ the matrix

$$\omega_\times = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \in \mathbb{R}^{3\times3}$$ is called the "left cross-product matrix"

and satisfies : $\omega_\times \cdot \vec{v} = \vec{\omega}\times\vec{v}$ for any $\vec{v}\in\mathbb{R}^3$

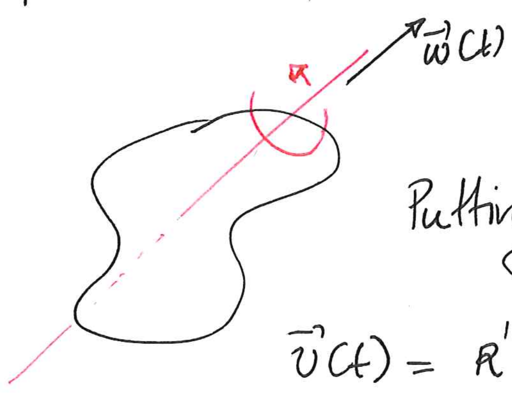
Theorem #2 Any 3×3 anti-symmetric matrix can be written as a cross-product matrix:

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} \\ a_{12} & 0 & -a_{23} \\ a_{13} & a_{23} & 0 \end{pmatrix} = u_\times \quad \text{where} \quad \vec{u} = \begin{pmatrix} a_{23} \\ -a_{13} \\ a_{12} \end{pmatrix}$$

Going back to our previous derivation, we write the anti-symmetric matrix $R'(t)\, R^T(t)$ as:

$$R'(t)\, R^T(t) = \omega_\times \quad \Rightarrow \quad \boxed{R'(t) = \omega_\times \cdot R(t)}$$

The vector $\vec{\omega}$ corresponding to the cross-product matrix $\omega_\times$ is called the angular velocity. It is aligned with the axis of rotation (use right-hand rule) and its magnitude measures the speed of rotation (in radians/sec)



Putting everything together, we get:

$$\vec{v}(t) = R'(t)\,\vec{x}_0 + \vec{v}_c(t)$$

$$= \omega_\times\, R(t)\,\vec{x}_0 + \vec{v}_c(t)$$

$$= \omega_\times \left\{ \vec{x}(t) - \vec{x}_c(t) \right\} + \vec{v}_c(t)$$

$$\boxed{\vec{v}(t) = \vec{\omega}(t) \times \vec{r}(t) + \vec{v}_c(t)}$$

Where $\vec{r}(t) = \vec{x}(t) - \vec{x}_c(t)$ is the offset from the center of mass, at time t.

When we were focusing on individual particles, we wrote the <u>state</u> of a particle as $(\vec{x}, \vec{v})$, and the governing equations as:

$$x'(t) = v(t)$$
$$v'(t) = \frac{1}{m} f(t, \vec{x}, \vec{v})$$

For a rigid body its <u>state</u> includes all of the following

- The position $\vec{x}_c(t)$ and velocity $\vec{v}_c(t)$ of the center of mass
- The orientation $R(t)$
- The angular velocity $\vec{\omega}(t)$.

Before we write the governing equations for R & ω, we will address the issues of energy, momentum for rotating bodies.

For a single moving particle, its kinetic energy is:

$$K = \frac{1}{2} m v^2$$

In order to determine the total kinetic energy of a rotating (and moving) rigid body we have to sum (integrate) all the kinetic energies of all infinitesimal chunks $\rho \cdot dV$ (ρ=density)

$$K = \int_{\Omega} \frac{1}{2} \rho(\vec{x}) \vec{v}^2(\vec{x}) \, d\vec{x}$$

Ultimately, this integral can be expressed in terms of the state $(\vec{x}_c, \vec{v}_c, R(t), \vec{\omega}(t))$, since $v = \vec{\omega} \times \vec{r} + \vec{v}_c$.

After (omitted) derivations, we arrive at:

$$K(t) = \frac{1}{2} \int_\Omega \rho(x)\, v^2(x(t))\, dV =$$

$$= \underbrace{\frac{1}{2} M v_c^2(t)}_{\text{linear kinetic energy}} + \underbrace{\frac{1}{2} \vec{\omega}(t)^T I(t)\, \omega(t)}_{\text{Rotational kinetic energy.}}$$

Here $M = \int_\Omega \rho\, dV$ = total mass of body.

and $I(t)$ = The <u>inertia tensor</u> (more on this, later).
(3x3 matrix).

For a single body (point mass) the law $f = ma$ is, really, the conservation of energy/momentum, in hiding. See:

$$\underbrace{E}_{\text{total energy}} = \underbrace{U}_{\text{potential energy (gravity, elasticity, etc)}} + \underbrace{K}_{\text{kinetic energy}}$$

$$= U(\vec{x}) + K(\vec{v}) = U(\vec{x}) + \frac{1}{2} m v^2$$

In a closed system, energy is conserved over time, i.e:

$$\frac{d}{dt} E(t) = 0 \quad \Rightarrow$$

$$0 = \frac{d}{dt}E(t) + \frac{d}{dt}\left\{\frac{1}{2}mv^2\right\} =$$

$$= \underbrace{\frac{\partial E}{\partial x}}_{-f(x)} \cdot \underbrace{\frac{dx}{dt}}_{v(t)} + mv(t)v'(t) = -f\cdot\cancel{v} + m\cancel{v}v' = 0$$

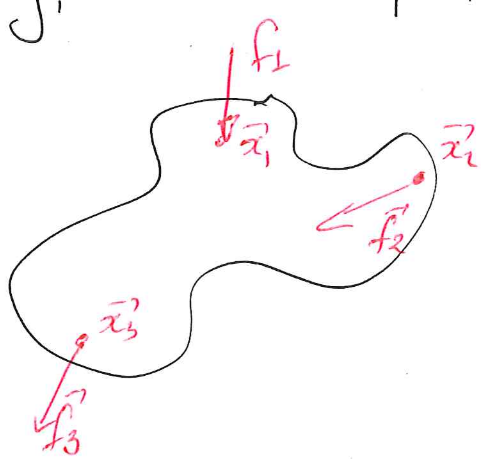
$$\Rightarrow \boxed{mv' = ma = f}$$

This last expression can also be written as:

$$\frac{d}{dt}\left[m\vec{v}(t)\right] = f \quad \text{or} \quad \frac{d}{dt}\vec{P}(t) = f$$

where $\vec{P}(t) = m\vec{v}(t)$ is the linear momentum.

In a rigid body, a number of external forces may be applied



The aggregate force $f_{tot}(t) = \Sigma f_i(t)$ satisfies the latter expression for the linear momentum of the body $\vec{P}(t) = M\cdot\vec{v}_c(t)$ i.e.

$$\frac{d}{dt}\left\{M\vec{v}_c(t)\right\} = f_{tot}(t) \quad \left(\text{or } v_c'(t) = \frac{1}{M}f(t)\right)$$

Similar to the linear momentum, a rotating body carries angular momentum as well, which is also a conserved quantity (in the absence of external forces). It is given by the formula:

$$\vec{L}(t) = I(t)\,\vec{\omega}(t)$$

The same set of forces $\{\vec{f_i}\}$, applied at locations $\{\vec{x}_i\}$, induce a torque

$$\vec{\tau} = \sum \left\{ \vec{x}_i(t) - \vec{x}_c(t) \right\} \times \vec{f}_i(t)$$

$$= \sum \vec{r}_i \times f_i \qquad \left[\vec{r}_i(t) := \vec{x}_i(t) - \vec{x}_c(t) \right].$$

The angular equivalent, now, of $f = ma$, is:

$$\boxed{\frac{d}{dt}\vec{L}(t) = \frac{d}{dt}\left\{ I(t)\,\vec{\omega}(t) \right\} = \vec{\tau}(t)} \qquad (*)$$

(Compare with $\frac{d}{dt}\{Mv(t)\} = f(t)$ ... in this equation we were able to factor out the time-independent mass M, but in $(*)$, the inertia tensor $I(t)$ has to stay inside the derivative).

In summary, the state variables of a rigid body are:

$$\begin{pmatrix} \vec{x}_c(t) \\ \vec{v}_c(t) \\ R(t) \\ \vec{\omega}(t) \end{pmatrix}$$

and the governing differential equations:

$$\vec{x}_c'(t) = v_c(t)$$

$$\vec{v}_c'(t) = \frac{1}{M} f(t) \quad \left(\text{this could be } f(t, \vec{x}_c, \vec{v}_c)\right)$$

or even $f(t, x_c, v_c, R, \omega)$

$$R'(t) = \omega_\times(t) R(t)$$

$$L'(t) = \frac{d}{dt}\left(I(t)\omega(t)\right) = \vec{\tau}(t) \quad \left(\text{or, } \tau(t, x_c, v_c, R, \omega)\right)$$

Notes:

→ When using forward Euler, equation $R'(t) = \omega_\times R(t)$ might produce a new matrix $R(t^{n+1})$ which is not a strict rotation.

e.g. $R(t^{n+1}) = R(t^n) + \Delta t\, \omega_\times(t^n) R(t^n)$

This can be fixed (somewhat) by re-projecting R^{n+1} to a rotation via the SVD: $R^{n+1} = U\Sigma V^T \Rightarrow R^{n+1}_{fixed} = UV^T$

(Alternatively, use quaternions. See later).

→ As we will see, next, the inertia tensor of a specific rigid body is a function of the current orientation $R(t)$, i.e $I(t) = I(R(t))$. Applying Forward Euler to the torque equation $\vec{L}'(t) = \vec{\tau}'(t)$, yields:

$$\vec{L}(t^{n+1}) = \vec{L}(t^n) + \Delta t\, \vec{\tau}(t^n)$$

$$I(R(t^{n+1}))\,\vec{\omega}(t^{n+1}) = I(R(t^n))\,\vec{\omega}(t^n) + \Delta t\, \tau(t^n)$$

$$\Rightarrow \quad \vec{\omega}(t^{n+1}) = \left[I(R^{n+1})\right]^{-1} \left\{ I(R^n)\,\vec{\omega}(t^n) + \Delta t\, \vec{\tau}(t^n) \right\}.$$

which necessitates that we <u>first</u> update the rotation R^{n+1}, and then the angular velocity $\vec{\omega}^{n+1}$

<u>The inertia tensor</u>

The inertia tensor $I(t)$ is computed as:

$$I(t) = \int_\Omega \rho(\vec{x}) \begin{pmatrix} y^2(t)+z^2(t) & -x(t)y(t) & -x(t)z(t) \\ -x(t)y(t) & x^2(t)+z^2(t) & -y(t)z(t) \\ -x(t)z(t) & -y(t)z(t) & x^2(t)+y^2(t) \end{pmatrix} dx\,dy\,dz$$

Thus, the value of $I(t)$ changes over time, and more specifically, changes with various orientations $R(t)$.

In fact we can show that

$$I(t) = R(t)\, I_0\, [R(t)]^T \qquad (**)$$

where I_0 is the inertia tensor computed in <u>body-space</u>.

<u>Notes</u>:

→ Due to equation (**), a popular approach is to pose a rigid body in body space so that, not only the center of mass is at the origin, but also I_0 = diagonal! We can do this by first picking an arbitrary orientation, and computing $\hat{I}_0$ (tentative). $\hat{I}_0$ is guaranteed to be symmetric, with an eigenanalysis $\hat{I}_0 = Q\, I_0\, Q^T$ diagonal

orthogonal

If we rotate this arbitrary configuration by Q^{-1} $(= Q^T)$, the inertia tensor would become the diagonal I_0!

→ There are various ways to compute (or approximate) the mass (M), center of gravity and inertia tensor of a body. If the shape is given as an equation or levelset, we can use Monte-Carlo integration, to approximate e.g. $\bar{x} = \frac{1}{N} \sum_i x_i$

If the rigid body is given as a triangulated boundary, see:

B. Mirtich "Fast & Accurate Comp. of Polyhedral Mass Properties" JGT 1(2) 1996.

→ To avoid issues with the update equation $R'(t) = \omega_\times R(t)$, we may choose to express the rotation as a <u>quaternion</u> $q(t) = (s(t), x(t), y(t), z(t))$.

If we do so, the update equation becomes:

$$\frac{d}{dt}\{q(t)\} = \frac{1}{2}\,\omega(t)\,q(t)$$

(where $\omega(t)\,q(t)$ denotes the multiplication between quaternions $[0, \vec{\omega}(t)]$ and $q(t)$].