THE UNIVERSITY of WISCONSIN MADISON

CS/ECE 552: Arithmetic and Logic

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Lecture notes based in part on slides created by Mark Hill, David Wood, Mikko Lipasti, Guri Sohi, John Shen and Jim Smith

Basic Arithmetic and the ALU

- Number representations: 2’s complement, unsigned
- Addition/Subtraction
- Add/Sub ALU
  - Full adder, ripple carry, subtraction
- Logical operations
  - and, or, xor, nor, shifts
- Overflow

Basic Arithmetic and the ALU

- Covered later in the semester:
  - Integer multiplication, division
  - Floating point arithmetic
- These are not crucial for the project

Background

- Recall
  - n bits enables 2^n unique combinations
- Notation: $b_{31}\ b_{30} \ldots b_3\ b_2\ b_1\ b_0$
- No inherent meaning
  - $f(b_{31} \ldots b_0)$ => integer value
  - $f(b_{31} \ldots b_0)$ => control signals

Background

- 32-bit types include
  - Unsigned integers
  - Signed integers
  - Single-precision floating point
  - MIPS instructions (refer to book)

Unsigned Integers

- $f(b_{31}...b_0) = b_{31} \times 2^{31} + ... + b_1 \times 2^1 + b_0 \times 2^0$
- Treat as normal binary number

  E.g. 0…01101010101

  $= 1 \times 2^7 + 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^1 + 1 \times 2^0$

  $= 128 + 64 + 16 + 4 + 1 = 213$
- Max $f(111...11) = 2^{32} – 1 = 4{,}294{,}967{,}295$
- Min $f(000...00) = 0$
- Range $[0,2^{32}-1]$ => # values $(2^{32}-1) – 0 + 1 = 2^{32}$

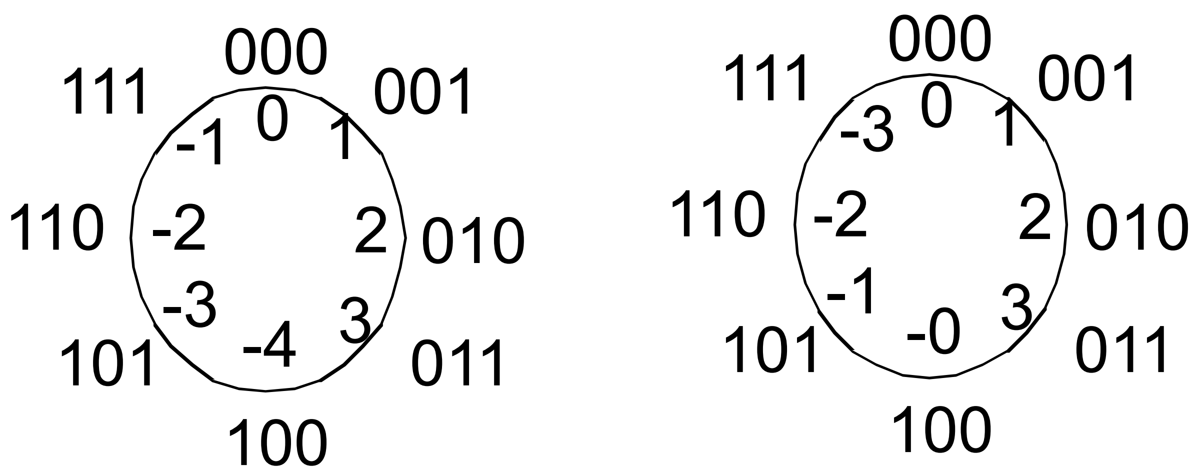
Signed Integers

- 2's complement

  $f(b_{31}...b_0) = -b_{31} \times 2^{31} + ... b_1 \times 2^1 + b_0 \times 2^0$
- Max $f(0111...11) = 2^{31} - 1 = 2147483647$
- Min $f(100...00) = -2^{31} = -2147483648$ (asymmetric)
- Range$[-2^{31}, 2^{31}-1]$ => # values$(2^{31}-1 - -2^{31}) = 2^{32}$
- Invert bits and add one: e.g. –6
  - 000…0110 => 111…1001 + 1 => 111…1010

Why 2's Complement

- Why not use sign-magnitude?
- 2's complement makes hardware simpler
- Just like humans don't work with Roman numerals
- Representation affects ease of calculation, not correctness of answer



Addition and Subtraction

- 4-bit unsigned example

0	0	1	1		3
1	0	1	0		10
1	1	0	1		13

- 4-bit 2's complement – ignoring overflow

0	0	1	1		3
1	0	1	0		-6
1	1	0	1		-3

Subtraction

- A – B = A + 2's complement of B
- E.g., 3 – 2

0	0	1	1		3
1	1	1	0		-2
0	0	0	1		1

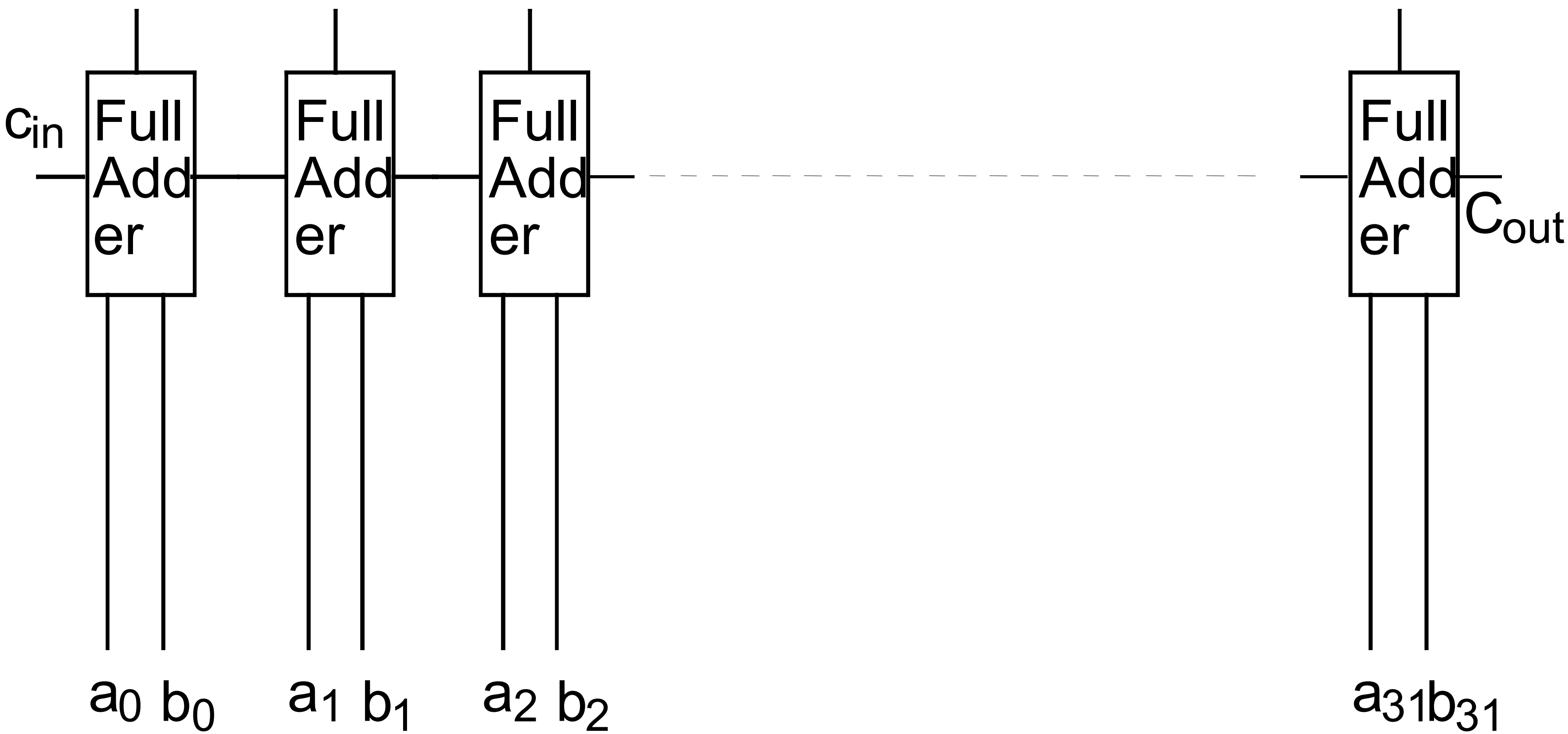
Full Adder

- Full adder $(a,b,c_{in}) \Rightarrow (c_{out}, s)$
- c_{out} = two or more of (a, b, c_{in})
- s = exactly one or three of (a,b,c_{in})

a	b	c_{in}	c_{out}	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

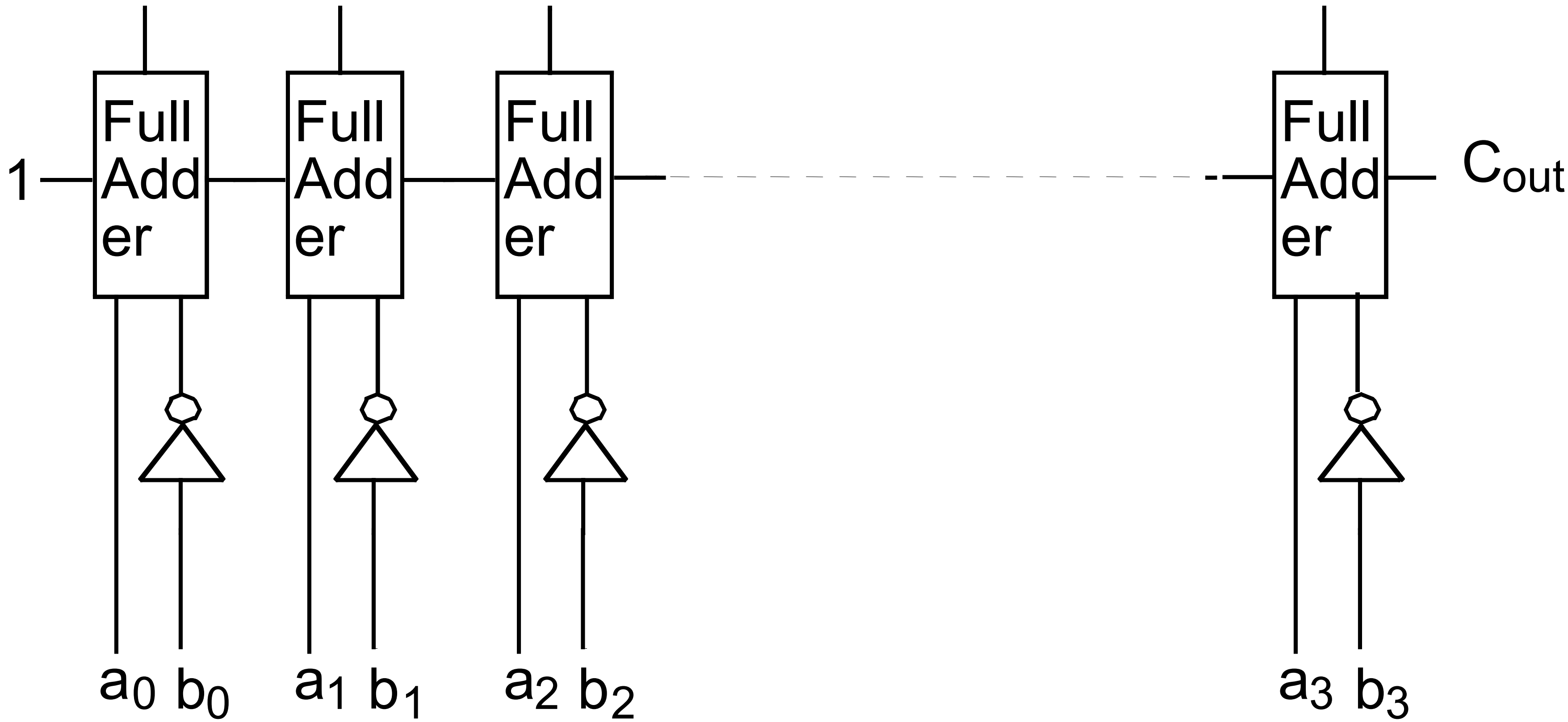
Ripple-carry Adder

- Just concatenate the full adders



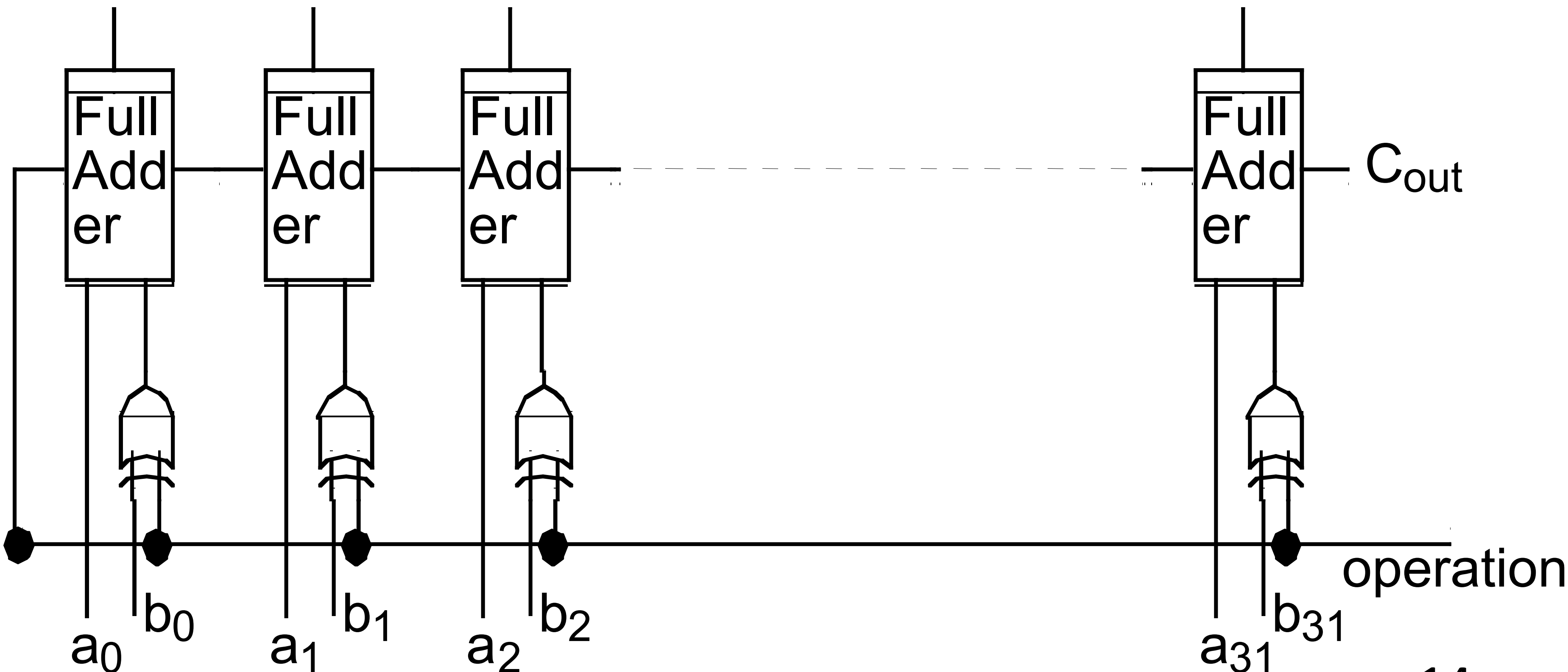
Ripple-carry Subtractor

- A – B = A + (-B) => invert B and set c_{in} to 1



Combined Ripple-carry Adder/Subtractor

- Control = 1 => subtract
- XOR B with control and set c_{in0} to control

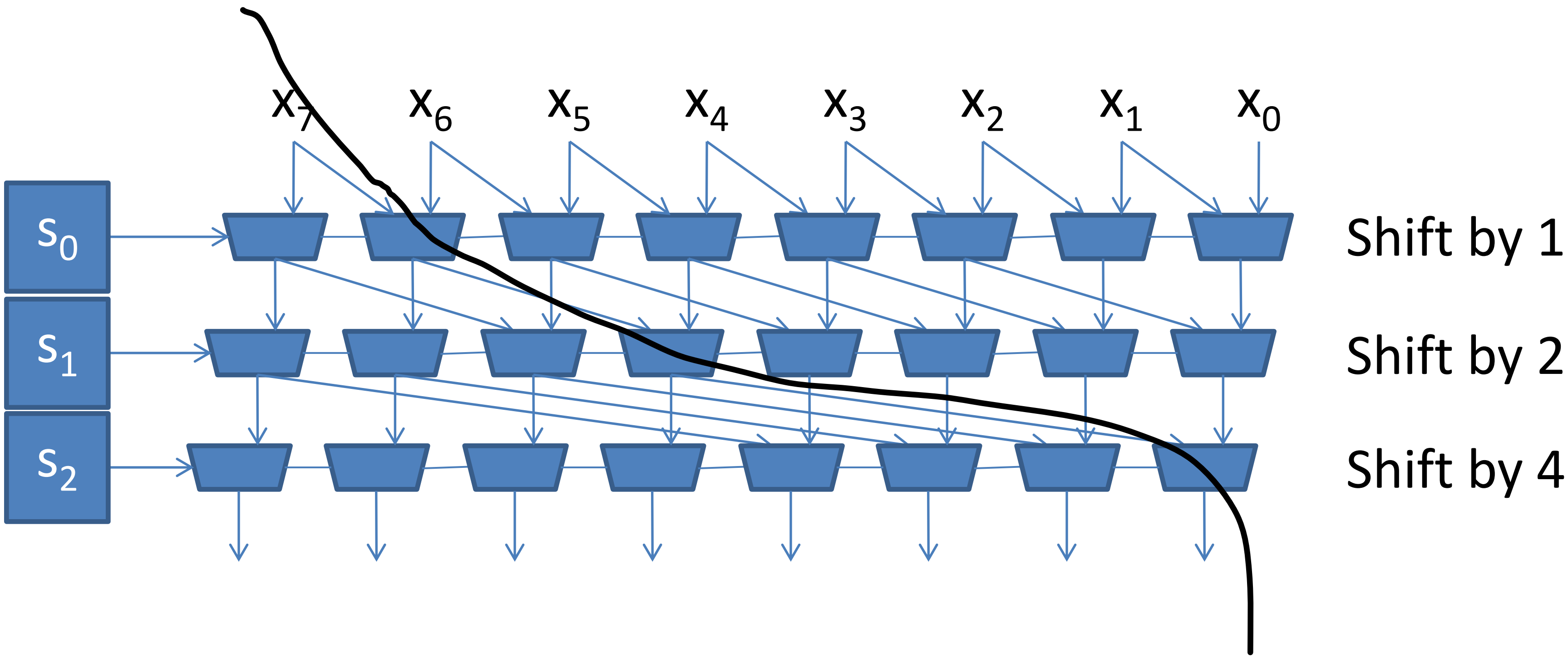


Logical Operations

- Bitwise AND, OR, XOR, NOR
  - Implement w/ 32 gates in parallel

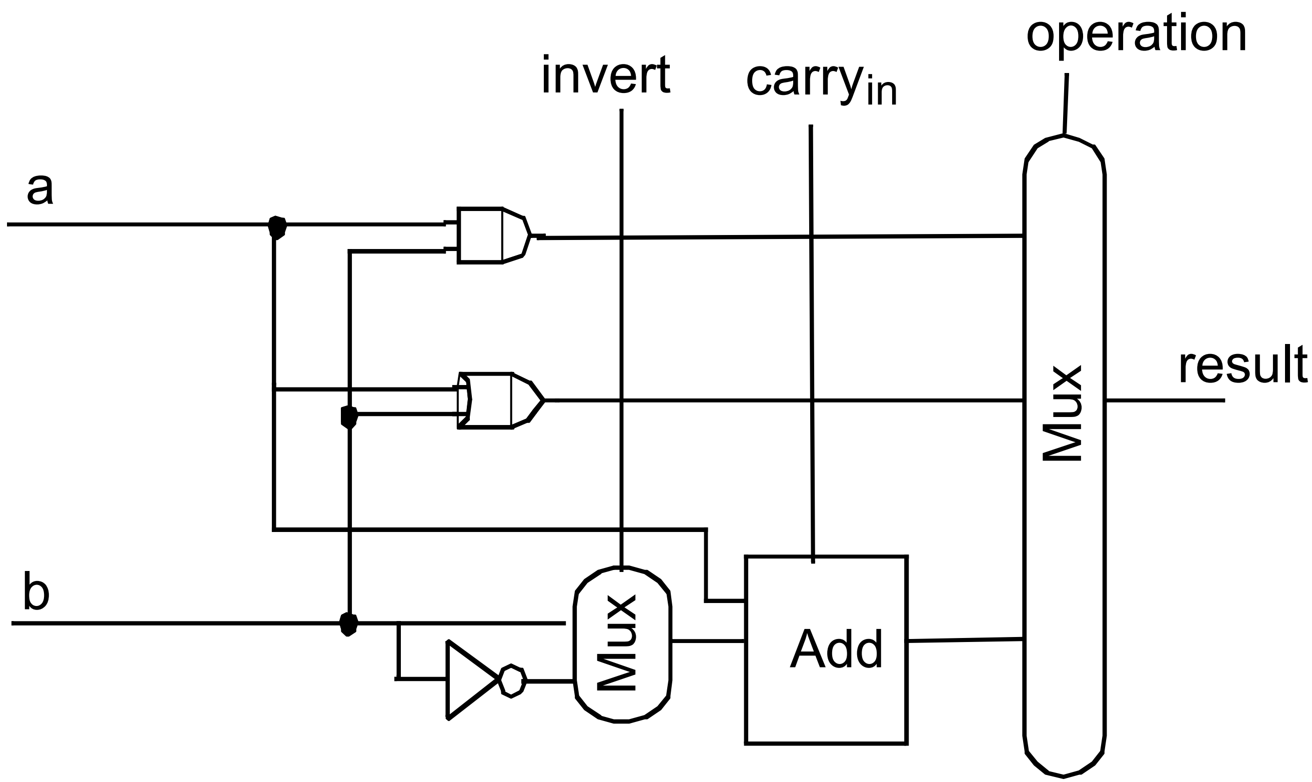
- Shifts and rotates
  - rol => rotate left (MSB->LSB)
  - ror => rotate right (LSB->MSB)
  - sll -> shift left logical (0->LSB)
  - srl -> shift right logical (0->LSB)
  - sra -> shift right arithmetic (old MSB->new MSB)

Shifter



- Right shift logic shown: missing inputs are 0
  - Left shift logic similar
- Rotate: wraparound instead of 0 inputs

All Together



Overflow

- With n bits only 2^n combinations
  - Unsigned range $[0, 2^n-1]$
  - 2's complement range $[-2^{n-1}, 2^{n-1}-1]$
- Unsigned Add

  5 + 6 > 7: 101 + 110 => 1011

  f(3:0) = a(2:0) + b(2:0) => overflow = f(3)

  Carryout from MSB

Overflow

- More involved for 2's complement

  -1 + -1 = -2:

  111 + 111 = 1110

  110 = -2 is correct
- Can't just use carry-out to signal overflow

Addition Overflow

- When is overflow NOT possible?

  $(p1, p2) > 0$ and $(n1, n2) < 0$

  $p1 + p2$

  $p1 + n1$ not possible

  $n1 + p2$ not possible

  $n1 + n2$
- Just checking signs of inputs is not sufficient

Addition Overflow

- 2 + 3 = 5 > 4: 010 + 011 = 101 =? –3 < 0
  - Sum of two positive numbers should not be negative
    - Conclude: overflow
- -1 + -4: 111 + 100 = 011 > 0
  - Sum of two negative numbers should not be positive
    - Conclude: overflow

Overflow = f(2) * ~(a2)*~(b2) + ~f(2) * a(2) * b(2)

Subtraction Overflow

- No overflow on a-b if signs are the same
- Neg – pos => neg ;; overflow otherwise
- Pos – neg => pos ;; overflow otherwise

Overflow = f(2) * ~(a2)*(b2) + ~f(2) * a(2) * ~b(2)

What to do on Overflow?

- Ignore ! (C language semantics)
  - What about Java? (try/catch?)
- Flag – condition code
- Sticky flag – e.g. for floating point
  - Otherwise gets in the way of fast hardware
- Trap – possibly maskable
  - MIPS has e.g. add that traps, addu that does not
  - Useful for extended precision in software

Zero and Negative

- Zero = ~[f(2) + f(1) + f(0)]
- Negative = f(2) (sign bit)

Zero and Negative

- May also want correct answer even on overflow
- Negative = (a < b) = (a – b) < 0 even if overflow
- E.g. is –4 < 2?

  100 – 010 = 1010 (-4 – 2 = -6): Overflow!

- Work it out: negative = f(2) XOR overflow

Summary

- Binary representations, signed/unsigned
- Arithmetic
  - Full adder, ripple-carry, adder/subtractor
  - Overflow, negative
- Logical
  - Shift, and, or
- Next: high-performance adders

– Later: multiply/divide/FP