MATHEMATICS OF OPERATIONS RESEARCH
Vol.19, No.4, November 1994, pp. 880-892.

NEW ERROR BOUNDS FOR THE LINEAR COMPLEMENTARITY PROBLEM*

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ABSTRACT

Recently [MaS93] it was shown that every nonlinear complementarity problem (NCP) is equivalent to the unconstrained minimization of a certain implicit Lagrangian. In particular, it was shown that this implicit Lagrangian is nonnegative everywhere and its set of zeros coincides with the solution set of the original NCP. In this paper, we consider the linear complementarity problem (LCP), and show that the distance to the solution set of the LCP from any point sufficiently close to the set can be bounded above by the square root of the implicit Lagrangian for the LCP. In other words, the square root of the implicit Lagrangian is a local error bound for the LCP. Our proof is based on showing that the square root of the implicit Lagrangian is equivalent to the residual function used in a known local error bound [Rob81, LuT92b]. When the matrix associated with the LCP is nondegenerate, the new error bound is in fact global. This extends the error bound result [MaP90] for the LCP with a P-matrix.

KEY WORDS. Error bound, linear complementarity problem.

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* The first author is supported by Natural Sciences and Engineering Research Council of Canada Grant OPG0090391. The last three authors are supported by Air Force Office of Scientific Research Grant AFOSR-89-0410 and National Science Foundation Grant CCR-9101801.

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1 Introduction

Consider the following nonlinear complementarity problem (NCP)

$$F(x) \geq 0, \quad x \geq 0, \quad \langle x, F(x) \rangle = 0, \tag{1.1}$$

where $F : \Re^n \mapsto \Re^n$ is a continuous mapping, $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product. We assume throughout that the solution set of the above NCP, denoted by P, is nonempty. Recently [MaS93] an interesting relation between the NCP (1.1) and the following implicit Lagrangian for (1.1) was established

$$M(x, \alpha) := 2\alpha\langle x, F(x)\rangle + \left(\|(-\alpha F(x) + x)_+\|^2 - \|x\|^2 + \|(-\alpha x + F(x))_+\|^2 - \|F(x)\|^2\right). \tag{1.2}$$

Here $(\cdot)_+$ denotes the orthogonal projection operator onto the nonnegative orthant $[0, \infty)^n$, $\|\cdot\|$ denotes the standard Euclidean 2-norm, and $\alpha > 0$ is a penalty parameter. Specifically, for each $\alpha > 1$, it was proven that the implicit Lagrangian $M(x, \alpha)$ is nonnegative for all x, and a point $x \in \Re^n$ solves the NCP (1.1) if and only if $M(x, \alpha) = 0$.

When $F(x)$ is an affine mapping, that is,

$$F(x) = Qx + q,$$

for some $n \times n$ matrix Q and some n-vector q, then (1.1) is called a linear complementarity problem (LCP). In this paper, we show that the distance to the solution set of the LCP from any point sufficiently close to the solution set can be bounded above by the square root of the implicit Lagrangian $M(x, \alpha)$ for the LCP. More precisely, we show that there exists some positive constants κ, δ (independent of x, α) such that

$$\operatorname{dist}(x, P) \leq \frac{\kappa}{\sqrt{\alpha - 1}}(M(x, \alpha))^{1/2}, \tag{1.3}$$

for all $\alpha > 1$ and all x with $M(x, \alpha) \leq \delta(\alpha - 1)$, where $\operatorname{dist}(\cdot, \cdot)$ denotes the distance between two sets, and P denotes the solution set of the LCP. Clearly, (1.3) implies that $x \in P$ whenever $M(x, \alpha) = 0$ for some $\alpha > 1$. The error bound (1.3) is *local* in the sense that it holds only for those x close enough to P (i.e., $M(x, \alpha)$ is sufficiently small). An error bound is *global* if it holds for all x.

Error bounds for LCP and related problems have been well studied (see [LuT92c], [Man85], [MaS87] and [MaP90]) and have found many important applications in the convergence analysis of some well known algorithms (see [Pan87], [LuT92a], [LuT92b]). These error bounds provide effective termination criteria for iterative algorithms (see [Pan86]) and can be used for sensitivity analysis when the problem data is subject to perturbation.

The paper is organized as follows. In Section 2, we discuss some basic properties of the implicit Lagrangian $M(x, \alpha)$. Section 3 contains the proof of the error bound (1.3) and an example which shows that it cannot hold globally in general. The principal tool used to prove (1.3) is given in Theorem 3.1 which shows the equivalence of $M(x,\alpha)^{1/2}$ with the natural residual $r(x)$ given in (2.3) for the NCP. In Section 4, we globalize the new error bound and show in Theorem 4.2 that for each LCP with a nondegenerate matrix and a nonempty solution set the error bound (1.3) holds globally for all $x \in \Re^n$. The latter result extends the error bound result of Mathias and Pang [MaP90]. In Section 5, we indicate some possible extensions. Concluding remarks are given in Section 6.

We briefly describe our notation. For a $k \times m$ matrix A, we will denote by A_i the i–th row of A and, for any nonempty $I \subseteq \{1, \ldots, k\}$, by A_I the submatrix of A obtained by removing all rows i of A such that $i \notin I$. Analogously, for any k-vector x and any nonempty subset $J \subseteq \{1, \ldots, k\}$, we denote by x_J the vector with components x_i, $i \in J$. Also, for any $J \subseteq \{1, \ldots, n\}$, we denote by $\tilde{J}$ the complement of J with respect to $\{1, \ldots, n\}$. Finally, we use the short notation $M'(x,\alpha)$ to denote the partial derivative of $M(x,\alpha)$ with respect to the parameter α.

2 Some Basic Properties of $M(x,\alpha)$

In this section, we shall derive some useful properties of the implicit Lagrangian $M(x,\alpha)$. These properties will be used later in Sections 3–4 to prove the desired error bound results.

Lemma 2.1. For all x in $\Re^n$: $M(x,1) = 0$.

Proof. Let

$$I = \{i \mid x_i \geq F_i(x)\}.$$

Then, it follows from (1.2) that

$$\begin{aligned} M(x,1) &= 2\langle x, F(x)\rangle - \|x\|^2 - \|F(x)\|^2 + \|x_I - F_I(x)\|^2 + \|F_{\tilde{I}}(x) - x_{\tilde{I}}\|^2 \\ &= 2\langle x, F(x)\rangle - \|x\|^2 - \|F(x)\|^2 + \left(\|x_I\|^2 - 2\langle x_I, F_I(x)\rangle + \|F_I(x)\|^2\right) \\ &\quad + \left(\|x_{\tilde{I}}\|^2 - 2\langle x_{\tilde{I}}, F_{\tilde{I}}(x)\rangle + \|F_{\tilde{I}}(x)\|^2\right) \\ &= 0. \end{aligned}$$

The proof is complete. **Q.E.D.**

Before stating the next lemma, we need to fix some notation. For any $(x,\alpha) \in \Re^n \times (0,\infty)$, we let

$$I_\alpha^x = \{i \mid 1 \leq i \leq n,\ x_i \geq \alpha F_i(x)\}, \quad J_\alpha^x = \{j \mid 1 \leq j \leq n,\ F_j(x) \geq \alpha x_j\} \tag{2.1}$$

and let

$$r_i(x) = x_i - (x_i - F_i(x))_+ = \min\{x_i, F_i(x)\} \tag{2.2}$$

be a component of the natural residual [Pan86, Lemma 2]

$$r(x) = \|x - (x - F(x))_+\|. \tag{2.3}$$

Lemma 2.2. Suppose that $\alpha > 1$ and $x \in \Re^n$. Then

$$x_i > 0, \quad F_i(x) > 0, \qquad \text{if } i \in \tilde{I}^x_\alpha \cap \tilde{J}^x_\alpha; \tag{2.4}$$

$$x_i \le 0, \quad F_i(x) \le 0, \qquad \text{if } i \in I^x_\alpha \cap J^x_\alpha; \tag{2.5}$$

$$r_i(x) = F_i(x), \qquad \text{if } i \in I^x_\alpha \cap \tilde{J}^x_\alpha; \tag{2.6}$$

$$r_i(x) = x_i, \qquad \text{if } i \in \tilde{I}^x_\alpha \cap J^x_\alpha. \tag{2.7}$$

Proof. Suppose $i \in \tilde{I}^x_\alpha \cap \tilde{J}^x_\alpha$. By the definition of I^x_α and J^x_α, we have

$$\alpha^2 x_i > \alpha F_i(x) > x_i \text{ and } \alpha^2 F_i(x) > \alpha x_i > F_i(x)$$

so that $(\alpha^2 - 1)x_i > 0$ and $(\alpha^2 - 1)F_i(x) > 0$. Since $\alpha > 1$, it follows $x_i > 0$ and $F_i(x) > 0$, as desired. Similarly, we can show that $x_i \le 0$ and $F_i(x) \le 0$ for all $i \in I^x_\alpha \cap J^x_\alpha$.

It remains to show (2.6) and (2.7). If $i \in I^x_\alpha \cap \tilde{J}^x_\alpha$, then $x_i \ge \alpha F_i(x)$ and $\alpha x_i > F_i(x)$. Thus, $x_i + \alpha x_i > \alpha F_i(x) + F_i(x)$, so $x_i > F_i(x)$. This implies that $r_i(x) = F_i(x)$. The relation (2.7) can be established similarly. **Q.E.D.**

From (1.2), we can see that $M(x, \alpha)$ is continuously differentiable with respect α, so $M'(x, \alpha)$ exists. The next lemma summarizes some elementary properties of $M'(x, \alpha)$.

Lemma 2.3. For each $(x, \alpha) \in \Re^n \times (0, \infty)$, there holds

$$M'(x, \alpha) = 2\langle x_{\tilde{I}\cap\tilde{J}}, F_{\tilde{I}\cap\tilde{J}}(x)\rangle + 2\alpha\|F_I(x)\|^2 + 2\alpha\|x_J\|^2 - 2\langle x_{I\cap J}, F_{I\cap J}(x)\rangle, \tag{2.8}$$

where $I = I^x_\alpha$ and $J = J^x_\alpha$. Moreover, if $\alpha > 1$, then

$$M'(x, \alpha) \ge \frac{1}{\alpha}\left(\|x_{\tilde{I}\cap\tilde{J}}\|^2 + \|F_{\tilde{I}\cap\tilde{J}}(x)\|^2\right) + \alpha\left(\|x_J\|^2 + \|F_I(x)\|^2\right). \tag{2.9}$$

Finally, $M'(x, \alpha) = 0$ for some $\alpha > 1$ if and only if $x \in P$.

Proof. By definition (cf. (1.2)), we have

$$\begin{aligned} M(x, \alpha) &= 2\alpha\langle x, F(x)\rangle + \left(\|(-\alpha F(x) + x)_+\|^2 - \|x\|^2 + \|(-\alpha x + F(x))_+\|^2 - \|F(x)\|^2\right) \\ &= 2\alpha\langle x, F(x)\rangle + \left(\|x_I - \alpha F_I(x)\|^2 - \|x\|^2 + \|F_J(x) - \alpha x_J\|^2 - \|F(x)\|^2\right) \end{aligned}$$

where the second step follows from the definition of I $(= I^x_\alpha)$ and J $(= J^x_\alpha)$. Thus, taking the derivative of both sides of the above equation with respect to α yields

$$\begin{aligned} M'(x, \alpha) &= 2\langle x, F(x)\rangle - 2\langle x_I, F_I(x)\rangle + 2\alpha\|F_I(x)\|^2 - 2\langle x_J, F_J(x)\rangle + 2\alpha\|x_J\|^2 \\ &= 2\langle x_{\tilde{I}\cap\tilde{J}}, F_{\tilde{I}\cap\tilde{J}}(x)\rangle + 2\alpha\|F_I(x)\|^2 + 2\alpha\|x_J\|^2 - 2\langle x_{I\cap J}, F_{I\cap J}(x)\rangle, \end{aligned}$$

which proves (2.8).

Suppose that $\alpha > 1$ and $i \in \tilde{I} \cap \tilde{J}$. Then, by Lemma 2.2, we have $x_i \geq 0$ and $F_i(x) \geq 0$. Moreover, since $\alpha F_i(x) \geq x_i$ for all $i \in \tilde{I}$, it follows that

$$\|x_{\tilde{I}\cap\tilde{J}}\|^2 \leq \alpha\langle x_{\tilde{I}\cap\tilde{J}}, F_{\tilde{I}\cap\tilde{J}}(x)\rangle.$$

Similarly, we have

$$\|F_{\tilde{I}\cap\tilde{J}}(x)\|^2 \leq \alpha\langle x_{\tilde{I}\cap\tilde{J}}, F_{\tilde{I}\cap\tilde{J}}(x)\rangle.$$

Combining the above two relations with (2.8) yields

$$M'(x,\alpha) \geq \frac{1}{\alpha}\left(\|x_{\tilde{I}\cap\tilde{J}}\|^2 + \|F_{\tilde{I}\cap\tilde{J}}(x)\|^2\right) + 2\alpha\|x_J\|^2 + 2\alpha\|F_I(x)\|^2 - 2\langle x_{I\cap J}, F_{I\cap J}(x)\rangle. \tag{2.10}$$

We now bound the last term in the above expression. Using Lemma 2.2 and the definition of I and J , we have for any $i \in I \cap J$

$$0 \geq x_i \geq \alpha F_i(x) \quad \text{and} \quad 0 \geq F_i(x) \geq \alpha x_i.$$

Thus, we can bound this term as follows

$$\langle x_{I\cap J}, F_{I\cap J}(x)\rangle \leq \alpha\|F_{I\cap J}(x)\|^2 \leq \alpha\|F_I(x)\|^2$$

and

$$\langle x_{I\cap J}, F_{I\cap J}(x)\rangle \leq \alpha\|x_{I\cap J}\|^2 \leq \alpha\|x_J\|^2.$$

Combining above two relations with (2.10) yields

$$M'(x,\alpha) \geq \frac{1}{\alpha}\left(\|x_{\tilde{I}\cap\tilde{J}}\|^2 + \|F_{\tilde{I}\cap\tilde{J}}(x)\|^2\right) + \alpha\left(\|x_J\|^2 + \|F_I(x)\|^2\right).$$

Finally, if $M'(x,\alpha) = 0$ with $\alpha > 1$, then we have from (2.9) that

$$x_{\tilde{I}\cap\tilde{J}} = F_{\tilde{I}\cap\tilde{J}}(x) = 0, \quad x_J = 0, \quad F_I(x) = 0.$$

Thus, we have $\langle x, F(x)\rangle = 0$. Also, we have from the definitions of I $(= I^x_\alpha)$ and J $(= J^x_\alpha)$ that

$$x_I \geq \alpha F_I(x), \quad x_{\tilde{I}} \leq \alpha F_{\tilde{I}}(x), \quad F_J(x) \geq \alpha x_J, \quad F_{\tilde{J}}(x) \leq \alpha x_{\tilde{J}}.$$

Combining the above two relations yields $x \geq 0$, $F(x) \geq 0$. This, together with $\langle x, F(x)\rangle = 0$, implies that $x \in P$. Conversely, if $x \in P$, then $x \geq 0$, $F(x) \geq 0$ and $\langle x, F(x)\rangle = 0$. Let $I = I^x_\alpha$ and $J = J^x_\alpha$. By (2.8), it can be easily verified that $M'(x,\alpha) = 0$, as desired. **Q.E.D.**

We remark that Lemmas 2.1–2.3 hold not just for LCP, but also for NCP, since we have not used the linearity of F in the proof of these lemmas. From these lemmas, we can easily infer the result of [MaS93]. In particular, for each $\alpha > 1$, we have from Lemma 2.1

$$\begin{aligned} M(x,\alpha) &= M(x,\alpha) - M(x,1) \\ &= \int_1^\alpha M'(x,\beta)d\beta \geq 0, \end{aligned}$$

where the last step follows from $M'(x,\beta) \geq 0$ (cf. Lemma 2.3). Therefore, $M(x,\alpha) = 0$ if and only if $M'(x,\beta) = 0$ for all $\beta \in [1,\alpha]$. By Lemma 2.3, the latter condition is further equivalent to $x \in P$. Thus, the zero set of $M(x,\alpha)$ coincides with the solution set P of the NCP. This recovers Theorem 2.1 of [MaS93].

3 A New Local Error Bound

In this section, we shall establish the desired local error bound (1.3). Our proof consists of first showing that the square root of the implicit Lagrangian is equivalent to the residual function $r(x)$ as defined by (2.3), and then using the following known local error bound (see [Rob81, LuT92b])

$$\mathrm{dist}(x,P) \leq \kappa r(x) \tag{3.1}$$

when $r(x) \leq \delta$. Here, δ and κ are two positive constants independent of x. To establish this equivalence, we need the following lemma.

Lemma 3.1. Let $\alpha > 1$. If $y,\, z \in \Re$ and $y \geq z$, then

$$2(\alpha-1)z^2 \leq 2\alpha zy + (z-\alpha y)_+^2 + (y-\alpha z)_+^2 - z^2 - y^2 \leq 2\alpha(\alpha-1)z^2. \tag{3.2}$$

Proof. Let $f := 2\alpha zy + (z-\alpha y)_+^2 + (y-\alpha z)_+^2 - z^2 - y^2$. We consider the following four cases.

Case 1. $(z-\alpha y) \leq 0$ and $(y-\alpha z) \geq 0$. Then, $f = (\alpha^2-1)z^2$. Since $\alpha > 1$, it follows that $2(\alpha-1)z^2 \leq f \leq 2\alpha(\alpha-1)z^2$, so (3.2) holds in this case.

Case 2. $(z-\alpha y) \geq 0$ and $(y-\alpha z) \geq 0$. Then, $f = (\alpha^2-1)z^2 + (z-\alpha y)^2$. Thus, for any fixed z, the minimum of f is $(\alpha^2-1)z^2$, which is no less than $2(\alpha-1)z^2$. On the other hand, since $z - \alpha y \geq 0$ and $y \geq z$, we have $\alpha y \leq z \leq y$. Hence, $(\alpha-1)y \leq 0$, which, by $\alpha > 1$, further implies $y \leq 0$. Now, since $z \leq y \leq z/\alpha$, the maximum of $(z-\alpha y)^2$ is $(z-\alpha z)^2 = (1-\alpha)^2 z^2$. Therefore, the maximum of f is $2\alpha(\alpha-1)z^2$, as desired.

Case 3. $(z-\alpha y) \leq 0$ and $(y-\alpha z) < 0$. Then, $f = 2\alpha zy - z^2 - y^2$. In this case, for fixed z, the maximum of f is $(\alpha^2-1)z^2$, which is bounded above by $2\alpha(\alpha-1)z^2$. On the other hand, we have $z \leq y < \alpha z$. So, it follows from $\alpha > 1$ that $y \geq z > 0$. Hence, for a fixed z, the minimum of f is $2(\alpha-1)z^2$, as desired.

Case 4. $(z-\alpha y) \leq 0$ and $(y-\alpha z) \geq 0$. This is impossible, since $\alpha y \leq z \leq y$ and $\alpha > 1$ would imply $y \leq 0$, which would further imply $y - \alpha z \geq 0$. This is a contradiction. **Q.E.D.**

We now use Lemma 3.1 to show that the residual $r(x)$ is equivalent to $(M(x,\alpha))^{1/2}$. More precisely, we have the following.

Theorem 3.1. For each $\alpha > 1$, the following holds

$$2(\alpha-1)r^2(x) \le M(x,\alpha) \le 2\alpha(\alpha-1)r^2(x), \qquad \forall x \in \Re^n. \tag{3.3}$$

Proof. For each $x \in \Re^n$, let

$$M_i(x,\alpha) := 2\alpha x_i F_i(x) + (-\alpha F_i(x) + x_i)_+^2 - x_i^2 + (-\alpha x_i + F_i(x))_+^2 - F_i(x)^2.$$

Then, we have

$$M(x,\alpha) = \sum_{i=1}^n M_i(x,\alpha).$$

Also, it follows from (2.2) that

$$r^2(x) = \sum_{i=1}^n (r_i(x))^2.$$

So, in order to prove (3.3), it is sufficient to show that

$$2(\alpha-1)(r_i(x))^2 \le M_i(x,\alpha) \le 2\alpha(\alpha-1)(r_i(x))^2, \qquad \forall i. \tag{3.4}$$

To this end, we only need to apply Lemma 3.1. In particular, let $z = \min\{x_i, F_i(x)\}$ and $y = \max\{x_i, F_i(x)\}$. Then, it can be verified that $y \ge z$ and $M_i(x,\alpha) = 2\alpha zy + (z-\alpha y)_+^2 + (y-\alpha z)_+^2 - z^2 - y^2$. Now (3.4) follows directly from (3.2). **Q.E.D.**

There are several consequences of Theorem 3.1. One consequence is that $(M(x,\alpha))^{1/2}$ can be used as a local upper bound for $\mathrm{dist}(x,P)$. This is because, by (3.1), $r(x)$ is a local upper bound for $\mathrm{dist}(x,P)$, and, by Theorem 3.1, $(M(x,\alpha))^{1/2}$ is equivalent to $r(x)$.

Theorem 3.2. There exist some positive constants κ, δ (independent of x and α) such that

$$\mathrm{dist}(x,P) \le \frac{\kappa}{\sqrt{\alpha-1}}(M(x,\alpha))^{1/2}, \tag{3.5}$$

for all $\alpha > 1$ and all x with $M(x,\alpha) \le \delta(\alpha-1)$.

The error bound (3.5) is a local one. The following example indicates that it cannot hold globally in general.

Example 3.1. Let

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $F(x) = Qx + q$. It can be easily verified that the corresponding LCP (1.1) has a unique solution $x^* = (0,0)^T$. Let $x^t = (0,t)^T$, where $t \ge 0$ is a parameter. Simple algebra yields

$$F(x^t) = Qx^t + q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\|(-\alpha F(x^t)+x^t)_+\|^2=(t-\alpha)^2, \qquad \|(-\alpha x^t+F(x^t))_+\|^2=1, \qquad \forall t\geq 1/\alpha.$$

Thus, we have $M(x^t,\alpha)=\alpha^2-1$, which remains bounded as $t\to\infty$. On the other hand, we have $\mathrm{dist}(x^t,P)=\|x^t\|=t$, which tends to infinity as $t\to\infty$. Thus, the error bound (3.5) cannot hold for the sequence $\{x^t\}$. In other words, (3.5) is not a global error bound in general.

Another consequence of Theorem 3.1 has to do with the globalization of the error bound (3.5). In particular, in [Man92] it has recently been shown that, for the class of nondegenerate monotone LCPs (i.e., Q is positive semi-definite and there exists some $x^*\in P$ such that $x^*+Qx^*+q>0$), $(1+\|x\|)r(x)$, or by a slight change of the argument $\max\{1,\|x\|\}r(x)$, can be used as a global upper bound for $\mathrm{dist}(x,P)$. In other words, even though $r(x)$ cannot be a global upper bound by itself, it becomes so if the extra factor $\max\{1,\|x\|\}$ is added. Since $r(x)$ is equivalent to $(M(x,\alpha))^{1/2}$, it follows that $\max\{1,\|x\|\}\,(M(x,\alpha))^{1/2}$ is also a global upper bound for $\mathrm{dist}(x,P)$.

Corollary 3.1. For each monotone nondegenerate LCP, there exists some constant $\kappa>0$ (independent of x and α) such that

$$\mathrm{dist}(x,P)\leq\frac{\kappa\max\{1,\|x\|\}}{\sqrt{\alpha-1}}\,(M(x,\alpha))^{1/2}, \qquad \forall\alpha>1,\ \forall x\in\Re^n.$$

Proof. Use the equivalence of $r(x)$ and $(M(x,\alpha))^{1/2}$ (Theorem 3.1). **Q.E.D.**

In Section 4, we shall strengthen Corollary 3.1 by removing monotonicity and nondegeneracy assumption (see Theorem 4.1). In other words, $\max\{1,\|x\|\}\,(M(x,\alpha))^{1/2}/\sqrt{\alpha-1}$ is a global upper bound for $\mathrm{dist}(x,P)$ as long as the LCP has a solution (i.e., P is nonempty).

4 Global Error Bounds

In this section we shall study the question of when the error bound (3.5) becomes global. Such question is of interest since the global error bound often leads to global linear rate of convergence when applied to the analysis of iterative algorithms (see [LuT92c]). We start with a globalization result which says that if an extra factor $\max\{1,\|x\|\}$ is added to the right hand side of the error bound (3.5), then it becomes global.

Theorem 4.1. Suppose P is nonempty. Then, there exists some positive constant τ (independent of x, α) such that

$$\mathrm{dist}(x,P)\leq\frac{\tau\max\{1,\|x\|\}}{\sqrt{\alpha-1}}\,(M(x,\alpha))^{1/2}, \qquad \forall\alpha>1,\ \forall x\in\Re^n. \tag{4.1}$$

Proof. Fix $x^*\in P$. Let δ and κ be the positive constants given by Theorem 3.2 and let $x\in\Re^n$ be any vector and α be any scalar greater than 1. We consider two cases.

Case 1. $M(x,\alpha)/\alpha - 1 \le \delta$. It follows from Theorem 3.2 that

$$\operatorname{dist}(x,P) \le \frac{\kappa}{\sqrt{\alpha-1}} \left(M(x,\alpha)\right)^{1/2} \le \frac{\kappa \max\{1,\|x\|\}}{\sqrt{\alpha-1}} \left(M(x,\alpha)\right)^{1/2}.$$

Case 2. $M(x,\alpha)/\alpha - 1 \ge \delta$. Then, we have

$$\begin{aligned}\operatorname{dist}(x,P) &\le \|x - x^*\| \\ &\le \|x\| + \|x^*\| \\ &\le (1+\|x^*\|)\max\{1,\|x\|\} \\ &\le \frac{1+\|x^*\|}{\sqrt{\delta(\alpha-1)}} \max\{1,\|x\|\} \left(M(x,\alpha)\right)^{1/2}.\end{aligned}$$

Combining this with the relation in Case 1 yields the desired global error bound (4.1). **Q.E.D.**

Compared to Corollary 3.1, Theorem 4.1 is stronger since it requires neither monotonicity nor nondegeneracy. As indicated by Example 3.1, the error bound (3.5) cannot be global in general, if the extra factor $\max\{1,\|x\|\}$ is not added. Can the error bound (3.5) hold globally for some restricted class of LCPs? What is the most general class of LCPs for which (3.5) holds globally? In the remainder of this section we shall show that the error bound (3.5) holds globally for the class of LCPs with a nondegenerate matrix. We need to make a definition.

Definition 4.1. An $n \times n$ matrix Q is called *nondegenerate* if all of its principal minors are nonzero.

For a discussion of such matrices see the recent book by Cottle, Pang and Stone [CPS92]. Recall that the class of P-matrices consists of those square matrices with positive principal minors. Thus, the class of nondegenerate matrices contains the class of P-matrices and hence the class of positive definite matrices.

We now state and prove our global error bound result.

Theorem 4.2. Let $F(x) = Qx + q$ where Q is a nondegenerate matrix. Then there exists a positive constant μ (independent of α, x) such that

$$\operatorname{dist}(x,P) \le \frac{\mu\sqrt{\alpha}}{\sqrt{\alpha-1}} \left(M(x,\alpha)\right)^{1/2}, \qquad \forall \alpha > 1,\ \forall x \in \Re^n. \tag{4.2}$$

Proof. We first note that (4.2) holds for any $\mu \ge \kappa$, any $\alpha > 1$ and all $x \in \{x \mid M(x,\alpha) \le \delta(\alpha-1)\}$ where κ and δ are defined in Theorem 3.2. Also, by Theorem 4.1, it is easy to see that (4.2) holds for any bounded set, provided μ is chosen appropriately. Hence it is enough to consider $x \in S$ where

$$S = \{x \mid M(x,\alpha) \ge \delta(\alpha-1), \|x\| \ge r\} \quad \text{and} \quad r = \max\left\{\frac{2\|q\|}{c},\ \|x^*\|,\ 1\right\} > 0. \tag{4.3}$$

Here x^* is any solution of the LCP and constant c will be defined later.

Let $I = I_\beta^x$ and $J = J_\beta^x$. Since, by Lemma 2.1,

$$M(x,\alpha) = M(x,\alpha) - M(x,1) = \int_1^\alpha M'(x,\beta)d\beta,$$

it follows from the Mean Value Theorem that there exists some $\beta \in (1,\alpha)$ such that $M(x,\alpha) = M'(x,\beta)(\alpha-1)$. Using (2.9) to bound $M'(x,\beta)$, we obtain

$$M(x,\alpha) = (\alpha-1)M'(x,\beta) \geq \frac{\alpha-1}{\beta}\left(\|x_{\tilde{I}\cap\tilde{J}}\|^2 + \|F_{\tilde{I}\cap\tilde{J}}(x)\|^2\right) + \beta(\alpha-1)\left(\|x_J\|^2 + \|F_I(x)\|^2\right).$$

Making use of the Cauchy–Schwarz and triangle inequalities and monotonicity of the 2-norm, we further obtain

$$\begin{aligned}(M(x,\alpha))^{1/2} &\geq \frac{\sqrt{\alpha-1}}{2\sqrt{\beta}}\left(\|x_{\tilde{I}\cap\tilde{J}}\| + \|F_{\tilde{I}\cap\tilde{J}}(x)\|\right) + \frac{\sqrt{\beta(\alpha-1)}}{2}\left(\|x_J\| + \|F_I(x)\|\right) \\ &\geq \frac{\sqrt{\alpha-1}}{2\sqrt{\beta}}\left(\|A_{I,J}x\| - \|q\|\right),\end{aligned} \tag{4.4}$$

where

$$A_{I,J} = \begin{pmatrix} Q_{I\cup(\tilde{I}\cap\tilde{J})} \\ E_{(\tilde{I}\cap\tilde{J})\cup J}\end{pmatrix}$$

with E denoting the identity matrix.

Consider the following three possible cases.

Case 1. Either $I = \{1,\ldots,n\}$ or $J = \emptyset$. Then $I \cup (\tilde{I}\cap\tilde{J}) = \{1,\ldots,n\}$ and nonsingularity of Q implies that $A_{I,J}x \neq 0$.

Case 2. Either $I = \emptyset$ or $J = \{1,\ldots,n\}$. Then $(\tilde{I}\cap\tilde{J}) \cup J = \{1,\ldots,n\}$ and $Ex = x \neq 0$ implies that $A_{I,J}x \neq 0$.

Case 3. Neither of the two previous cases occur. Then neither of the sets I, $\tilde{I}$, J, $\tilde{J}$ is empty. Suppose first that $\tilde{I}\cap\tilde{J} \neq \emptyset$. Then by Lemma 2.2, $x_{\tilde{I}\cap\tilde{J}} > 0$. Hence $A_{I,J}x \neq 0$. Suppose now $\tilde{I}\cap\tilde{J} = \emptyset$. Then we have $\tilde{J} \subseteq I$. Suppose $A_{I,J}x = 0$. This is equivalent to

$$Q_I x = 0 \ \text{ and } \ x_J = 0.$$

Since $\tilde{J} \subseteq I$ we have $Q_{\tilde{J}\tilde{J}}x_{\tilde{J}} = 0$. But the latter contradicts the fact that all the principal submatrices of Q are nonsingular. Therefore $A_{I,J}x \neq 0$.

Combining the above three cases we conclude that $A_{I,J}$ has full column rank. Thus we have

$$\inf_{x\in\Re^n}\frac{\|A_{I,J}x\|}{\|x\|} = \inf_{\|x\|=1}\|A_{I,J}x\| = \|A_{I,J}\bar{x}\| = c_{I,J} > 0$$

where infimum is attained at some point $\bar{x}$ by the compactness of the unit sphere. Hence

$$\|A_{I,J}x\| \geq c_{I,J}\|x\| \qquad \forall x \in \Re^n. \tag{4.5}$$

Let

$$c = \min\{c_{I,J} \mid \forall\, I, J \subseteq \{1, \ldots, n\}\} > 0. \tag{4.6}$$

Then from (4.4)–(4.6) we have

$$(M(x,\alpha))^{1/2} \geq \frac{\sqrt{\alpha - 1}}{2\sqrt{\beta}}\left(c\|x\| - \|q\|\right).$$

Combining this with

$$\operatorname{dist}(x, P) \leq \|x - x^*\| \leq \|x\| + \|x^*\|,$$

we obtain

$$\begin{aligned}
\frac{(M(x,\alpha))^{1/2}}{\operatorname{dist}(x,P)} &\geq \frac{\sqrt{\alpha - 1}}{2\sqrt{\beta}}\frac{c\|x\| - \|q\|}{\|x\| + \|x^*\|} \\
&= \frac{\sqrt{\alpha - 1}}{2\sqrt{\beta}}\frac{c - \|q\|/\|x\|}{1 + \|x^*\|/\|x\|} \\
&\geq \frac{\sqrt{\alpha - 1}}{2\sqrt{\beta}}\frac{c/2}{2} \\
&\geq \frac{c\sqrt{\alpha - 1}}{8\sqrt{\alpha}} > 0,
\end{aligned}$$

where the second last step is a consequence of (4.3). Now we let $\mu = 8/c$ and the proof is complete. **Q.E.D.**

As an important corollary of Theorem 3.1 and Theorem 4.2, we have the following global error bound result.

Theorem 4.3. Suppose Q is a nondegenerate matrix. Then, there exists some positive constant τ such that

$$\operatorname{dist}(x, P) \leq \tau r(x), \qquad \forall x \in \Re^n. \tag{4.7}$$

Proof. Fix some $\alpha > 1$, and then use Theorem 4.2 and the equivalence of $r(x)$ and $(M(x,\alpha))^{1/2}$ (see Theorem 3.1). **Q.E.D.**

Theorem 4.3 strengthens the global error bound result by Mathias and Pang [MaP90] who showed that (4.7) holds when Q is a P-matrix. [Recall that any P-matrix is necessarily nondegenerate, but the converse is not true.] We remark that our proof of the global error bound result (4.7) is by way of the equivalence of $r(x)$ and $(M(x,\alpha))^{1/2}$ and then using the global error bound (4.2). It is not clear whether one can prove (4.7) directly.

5 Extensions

In this section we describe some extensions of the new error bound results given in the previous sections. For simplicity, the proofs are omitted since they are for the most part analogous to the ones given before.

5.1 Equivalence of Different Error Bounds

We start with the following equivalence result (compare with Theorem 3.1).

Theorem 5.1. For each $\alpha > 1$, there holds

$$2r^2(x) \leq M'(x, \alpha) \leq 2\alpha r^2(x), \qquad \forall x \in \Re^n.$$

The proof of Theorem 5.1 is analogous to that of Theorem 3.1, and is thus omitted. By Theorems 4.1 and 5.1, we see that, for each $\alpha > 1$, the quantities $r(x)$, $(M'(x,\alpha))^{1/2}$ and $(M(x,\alpha))^{1/2}$ are all equivalent. Several consequences of this equivalence follow.

(1) These three quantities can all be used as upper bounds for $\operatorname{dist}(x, P)$ locally around the solution set.

(2) These quantities are global upper bounds for $\operatorname{dist}(x, P)$ if and only if any one of them is. For any LCP with a nondegenerate matrix and nonempty solution set P, each of the three quantities is a global upper bound for $\operatorname{dist}(x, P)$. In particular, for each $\alpha > 1$, there exists some positive constant τ such that

$$\operatorname{dist}(x, P) \leq \tau \left(M'(x, \alpha)\right)^{1/2}, \qquad \forall x \in \Re^n.$$

(3) If an extra term $\max\{1, \|x\|\}$ multiplies to each of these three quantities, then they each become a global upper bound for $\operatorname{dist}(x, P)$.

5.2 Error Bounds Using the Restricted Implicit Lagrangian

In Theorem 2.5 of [MaS93] it was also shown that, for positive α, the zeros in the nonnegative orthant of the following restricted implicit Lagrangian

$$N(x, \alpha) = 2\alpha\langle x, F(x)\rangle + \left(\|(-\alpha F(x) + x)_+\|^2 - \|x\|^2\right) \tag{5.1}$$

coincide with the solutions of the NCP. Using entirely different arguments, Fukushima [Fuk92] also showed the same result for $\alpha = 1$. Similar to (2.1), we let

$$I_\alpha^x = \{i \mid 1 \leq i \leq n,\ x_i \geq \alpha F_i(x)\}$$

and use $N'(x,\alpha)$ to denote the partial derivative of $N(x,\alpha)$ with respect to the parameter α. It is possible to prove the following basic properties of $N(x,\alpha)$.

(i) For any $\alpha > 0$, $N(x,\alpha)$ is nonnegative on the nonnegative orthant, and is zero if and only if $x \in P$.

(ii) $N(x,0) = 0$ for all $x \in \Re^n_+$.

(iii) For any $\alpha > 0$ and $x \in \Re^n_+$, we have

$$N'(x,\alpha) = 2\langle x_{\tilde{I}}, F_{\tilde{I}}(x)\rangle + 2\alpha\|F_I(x)\|^2 \geq \frac{2}{\alpha}\|x_{\tilde{I}}\|^2 + 2\alpha\|F_I(x)\|^2, \tag{5.2}$$

where $I = I^x_\alpha$.

(iv) For any $\alpha > 0$ and $x \in \Re^n_+$, $N'(x,\alpha) = 0$ if and only if $x \in P$.

Using these properties and following an argument similar to that used in Section 3, we can establish the following local error bound results.

Theorem 5.2. There exist positive constants ν and η (independent of x and α) such that

$$\operatorname{dist}(x,P) \leq \frac{\nu}{\sqrt{\alpha}}(N(x,\alpha))^{1/2}, \quad \operatorname{dist}(x,P) \leq \nu(N'(x,\alpha))^{1/2} \tag{5.3}$$

for all $(\alpha, x) \in \Re_+ \times \Re^n_+$ with either $N(x,\alpha) \leq \eta\alpha$ or $N'(x,\alpha) \leq \eta$.

Moreover, we can establish global error bound results similar to those in Section 4. We need to make a definition.

Definition 5.1. An $n \times n$ matrix Q is said to be nondegenerate with respect to the nonnegative orthant if the system $Q_{II}x_I = 0$, $x_I \geq 0$ has no nonzero solution for any $I \subseteq \{1,\ldots,n\}$, where Q_{II} denotes the principal submatrix indexed by I.

The class of matrices which are nondegenerate with respect to the nonnegative orthant is quite large and contains many well known classes of matrices. For example, it contains the class of nondegenerate matrices, the class of P-matrices and the class of positive definite matrices. It also contains the class of strictly semimonotone matrices (see Theorem 3.9.11 of [CPS92] for the characterizations of such matrices).

Our result is the following.

Theorem 5.3. Let $F(x) = Qx + q$ and $\alpha > 0$. Then the following hold.
(a) There exists some positive constant τ (depending on α) such that

$$\operatorname{dist}(x,P) \leq \tau \max\{1, \|x\|\} \left(N(x,\alpha)\right)^{1/2}, \qquad \forall x \in \Re^n_+.$$

(b) Suppose Q is a nondegenerate matrix with respect to the nonnegative orthant. Then there exists a positive constant γ (depending on α) such that

$$\operatorname{dist}(x, P) \leq \gamma \left(N(x, \alpha)\right)^{1/2}, \qquad \forall x \in \Re^n_+.$$

6 Concluding Remarks

In this paper, we have established some new local/global error bounds for linear complementarity problem. These new error bounds are based on the implicit Lagrangian $M(x, \alpha)$ recently introduced and studied in [MaS93]. It is interesting to note that these error bounds hold for all choices of the penalty parameter α, so long as $\alpha > 1$. Although we have not tried to find an α that gives the tightest error bound, it is possible to determine such optimal α by minimizing $M(x, \alpha)/(\alpha - 1)$ (see Theorem 3.2). Finally, it remains to be seen if this new error bound can be used to devise new iterative algorithms for solving LCP, or be used to analyze the convergence of the existing algorithms.

Acknowledgement: We are grateful to one of the referees for suggesting Lemma 3.1 which has led to a significant simplification of the proof of Theorem 3.1.

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