525 FINAL EXAM FALL 2009

1

	x_1	x_2	x_3	1
$x_4 =$	2	0	1	-2
$x_5 =$	0	3	1	-1
$z =$	4	-12	-1	0

Scheme II:

	x_1	x_2	x_5	1
$x_4 =$	2	-3	1	-1
$x_3 =$	0	-3	1	1
$z =$	4	-9	-1	-1

delete x_5 column

	x_1	x_2	1
$x_4 =$	2	-3	-1
$x_3 =$	0	-3	1
$z =$	4	-9	-1

	x_4	x_2	1
$x_1 =$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
$x_3 =$	0	-3	1
$z =$	2	-9	1

move x_1 to bottom

	x_4	x_2	1
$x_3 =$	0	-3	1
$z =$	2	-9	1
$x_1 =$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$

	x_4	x_3	1
$x_2 =$	0	$-\frac{1}{3}$	$\frac{1}{3}$
$z =$	2	3	-2
$x_1 =$	$\frac{1}{2}$	$-\frac{1}{2}$	1

solution: $x = \begin{pmatrix} 1 \\ 1/3 \\ 0 \end{pmatrix}$

(2)

	x_1	x_2	1
$x_3 =$	-1	1	1
$x_4 =$	0	-1	3
$z =$	-1	0	0
$z_0 =$	1	1	0

optimal for $-1+t \geq 0$, $t \geq 0 \Rightarrow t \geq 1$
with solution $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and solution $z(t) = 0$

Pivot on (1,1):

	x_3	x_2	1
$x_1 =$	-1	1	1
$x_4 =$	0	-1	3
$z =$	1	-1	-1
$z_0 =$	-1	2	1

optimal for $1-t \geq 0$, $-1+2t \geq 0 \Rightarrow t \in [\frac{1}{2}, 1]$
with solution $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $z(t) = -1+t$

Pivot on (2, 2)

$$\begin{array}{r|cc|c|} & x_3 & x_4 & 1 \\ \hline x_1 = & -1 & -1 & 4 \\ x_2 = & 0 & -1 & 3 \\ \hline z = & 1 & 1 & -4 \\ z_0 = & -1 & -2 & 7 \\ \hline \end{array}$$

optimal for $1-t \geq 0$, $1-2t \geq 0 \Rightarrow t \leq \frac{1}{2}$

with solution $x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $z(t) = -4+7t$

t	$x(t)$	$z(t)$
$(-\infty, \frac{1}{2}]$	$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$	$-4+7t$
$[\frac{1}{2}, 1]$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$-1+t$
$[1, \infty)$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	0

③ (a)
$$0 \leq 2x_1 + 2x_2 - u_1 - u_2 + 1 \perp x_1 \geq 0$$
$$0 \leq 2x_1 + 4x_2 - 2u_1 + u_2 \perp x_2 \geq 0$$
$$0 \leq x_1 + 2x_2 - 3 \perp u_1 \geq 0$$
$$0 \leq x_1 - x_2 + 2 \perp u_2 \geq 0$$

(b) $z = \begin{pmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{pmatrix}$ $M = \begin{bmatrix} 2 & 2 & -1 & -1 \\ 2 & 4 & -2 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$ $q = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}$

Phase I

	z_1	z_2	z_3	z_4	z_0	1
$w_1 =$	2	2	-1	-1	1	1
$w_2 =$	2	4	-2	1	1	0
$w_3 =$	1	2	0	0	1	-3
$w_4 =$	1	-1	0	0	1	2

	z_1	z_2	z_3	z_4	w_3	1
$w_1 =$	1	0	-1	-1	1	4
→ $w_2 =$	1	2	-2	1	1	3
$z_0 =$	-1	-2	0	0	1	3
$w_4 =$	0	-3	0	0	1	5

	z_1	z_2	w_2	z_4	w_3	1
$w_1 =$	$\frac{1}{2}$	-1	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$5/2$
$z_3 =$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$3/2$
→ $z_0 =$	-1	-2	0	0	1	3
$w_4 =$	0	-3	0	0	1	5

	z_1	z_0	w_2	z_4	w_3	1
$w_1 =$						1
$z_3 =$			…			3
$z_2 =$						$\frac{3}{2}$
$w_4 =$						$\frac{1}{2}$

solution $z = \begin{pmatrix} 0 \\ 3/2 \\ 3 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix} \quad u = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

(c). First KKT condition has

$$2x_1 + 2x_2 - u_1 - u_2 + 1 = 1$$
$$x_1 = 0.$$

(coefficient of x_1 does not appear in the KKT conditions)

if we increase 1 to 4 we have for the <u>same solution</u> x, u that

$$2x_1 + 2x_2 - u_1 - u_2 + 4 = 4$$
$$x_1 = 0$$

so the first KKT condition is still satisfied. The others are still satisfied too, so $x = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}$ $u = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is <u>still</u> <u>optimal</u>.

(4). $Ax > 0$ for some x if and only if $Ax \geq e$ for some x, where $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

(just scale x by a positive constant so that the smallest entry of Ax is 1)

Consider the following primal-dual pair:

P: $\min\ 0'x$ s.t. $Ax \geq e$

D: $\max\ e'u$ st $A'u = 0$, $u \geq 0$

If statement I is true then (P) is feasible, and hence optimal, since its objective is bounded below. Strong duality ⇒ (D) is optimal also, with optimal objective $e'u^* = 0$. If statement II were true, there would exist u satisfying the constraints of (D) with $e'u > 0$, contradicting optimality of u^*. Hence statement II cannot be true

If statement I is false, then (P) is infeasible. Hence by strong duality (D) is either infeasible or unbounded. It is not infeasible, since $u = 0$ clearly satisfies the constraints of (D). Hence it's unbounded, so in particular there is some u with $A'u = 0$, $u \geq 0$, $e'u > 0$. that is, not all components of u are zero, so statement II is true

5 (a) $A = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}$

(b). loss for wife $= x^T A y$

$\bar{x}' A \bar{y} = 1$,

$x' A \bar{y} = x' \begin{pmatrix} 1 \\ 4 \end{pmatrix} = x_1 + 4x_2 = 1 + 3x_2 \geq$

$= x_1 + 4x_2$

$= 1 + 3x_2$

$\geq 1 = \bar{x}' A \bar{y}$

$$\bar{x}' B \bar{y} = 2$$

$$\bar{x}' B y = \begin{pmatrix} 2 \\ 4 \end{pmatrix}' \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= 2y_1 + 4y_2$$

$$= 2 + 2y_2$$

$$\geq 2 = \bar{x}' B \bar{y}$$

so $\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\bar{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is Nash eq.

(c) $$\bar{x}' A \bar{y} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}' \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}' \begin{pmatrix} 14/5 \\ 14/5 \end{pmatrix}$$

$$= \frac{14}{5}$$

$$x' A \bar{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{pmatrix} 14/5 \\ 14/5 \end{pmatrix}$$

$$= 14/5 = x' A \bar{y}$$

$$\bar{x}' B \bar{y} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}' \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{14}{5} & \frac{14}{5} \end{pmatrix} \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = \frac{14}{5}$$

$$\bar{x}' B y = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}' \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{14}{5} & \frac{14}{5} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \frac{14}{5} = \bar{x}' B \bar{y}.$$