CS 525 FINAL SPRING 2009.

1. (a) $p = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$ $b = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

primal: $\min\ p'x$ s.t. $Ax \geq b$, $x \geq 0$

dual $\max\ b'u$ s.t. $A'u \leq p$, $u \geq 0$

i.e. $\max\ 4u_1 + 10u_2$ s.t. $u_1 + 3u_2 \leq 2$

$-u_1 + 2u_2 \leq -1$

3

(b)

		$u_3 =$ x_1	$u_4 =$ x_2	$w =$ 1
$-u_1$	$x_3 =$	1	-1	-4
$-u_2$	$x_4 =$	3	2	-10
1	$z =$	2	-1	0

Phase I:

		$u_3 =$ x_1	$u_4 =$ x_2	x_0	$w =$ 1
$-u_1$	$x_3 =$	1	-1	1	-4
$-u_2$	$x_4 =$	3	2	1	-10
1	$z =$	2	-1	0	0
	$z_0 =$	0	0	1	0

Special Pivot.

		$u_3 =$ x_1	$u_4 =$ x_2	$u_2 =$ x_4	$w =$ 1
$-u_1$	$x_3 =$	-2	-3	1	6
	$x_0 =$	-3	-2	1	10
1	$z =$	2	-1	0	0
	$z_0 =$	-3	-2	1	10

3

$$
\begin{array}{cc|ccc|c}
 & & u_1 = x_3 & u_4 = x_2 & u_2 = x_4 & w = 1 \\
\hline
-u_3 & x_1 = & -1/2 & -3/2 & 1/2 & 3 \\
 & x_0 = & 3/2 & 5/2 & -1/2 & 1 \\
\hline
1 & z = & -1 & -4 & 1 & 6 \\
 & z_0 = & 3/2 & 5/2 & -1/2 & 1 \\
\hline
\end{array}
$$

$$
\begin{array}{cc|ccc|c}
 & & u_1 = x_3 & u_4 = x_2 & x_0 & w = 1 \\
\hline
-u_3 & x_1 = & 1 & 1 & -1 & 4 \\
-u_2 & x_4 = & 3 & 5 & -2 & 2 \\
\hline
1 & z = & 2 & 1 & -2 & 8 \\
 & \cancel{z_0 =} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\
\hline
\end{array}
$$

(4) end of Phase I, delete row and col to get

$$
\begin{array}{cc|cc|c}
 & & u_1 = x_3 & u_4 = x_2 & w = 1 \\
\hline
-u_3 & x_1 = & 1 & 1 & 4 \\
-u_2 & x_4 = & 3 & 5 & 2 \\
\hline
1 & z = & 2 & 1 & 8 \\
\hline
\end{array}
$$

optimal! Solutions are

(2) $x^* = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad u^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$

optimal objective $z^* = w^* = 8$.

(c) Dual constraints not affected by the modification, so u^* is feasible for modified dual, with

(5) modified dual objective $10u_1^* + 10u_2^* = 20$.

By weak duality this is a lower bd on the modified primal.

1c(d) No change to solution, since x^* already satisfies the tighter constraints.

(3)

2. QP is
$$\min_x \ \frac{1}{2}\begin{pmatrix}x_1\\x_2\end{pmatrix}'\begin{pmatrix}1 & -2\\-2 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0\\4\end{pmatrix}'\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$\text{s.t.} \quad -x_1-2x_2 \geq -3$$
$$-2x_1-x_2 \geq -4$$
$$x_1, x_2 \geq 0.$$

(3)

LCP:
$$0 \leq Qx+p-A'u \perp x \geq 0$$
$$0 \leq Ax-b \perp u \geq 0.$$

$$\begin{pmatrix}z_1\\z_2\\z_3\\z_4\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\u_1\\u_2\end{pmatrix}$$

	z_1	z_2	z_3	z_4	1
$w_1 =$	1	-2	1	-2	0
$w_2 =$	-2	4	2	1	4
$w_3 =$	-1	-2	0	0	3
(3) $w_4 =$	2	-1	0	0	-4

1

Phase I :

	z_1	z_2	z_3	z_4	z_0	1
$w_1 =$	1	-2	1	-2	1	0
$w_2 =$	-2	4	2	1	1	4
$w_3 =$	-1	-1	0	0	1	3
(3) $w_4 =$	2	-1	0	0	1	-4

	z_1	z_2	z_3	z_4	w_4	1
$w_1 =$	-1	-1	1	-2	1	4
$w_2 =$	-4	5	2	1	1	8
$w_3 =$	-3	0	0	0	1	7
$z_0 =$	-2	1	0	0	1	4

	z_1	z_2	z_3	w_1	w_4	1
$z_4 =$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	2
$w_2 =$	$-\frac{9}{2}$	$\frac{9}{2}$	$\frac{5}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	10
$w_3 =$	-3	0	0	0	1	7
$z_0 =$	-2	1	0	0	1	4

$$
\begin{array}{r|ccccc|c}
 & z_0 & z_2 & z_3 & w_1 & w_4 & 1 \\
\hline
z_4 = & \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{4} & 1 \\
w_2 = & \frac{9}{4} & \frac{9}{4} & \frac{5}{2} & \frac{-1}{2} & \frac{-3}{4} & 1 \\
w_3 = & \frac{3}{2} & \frac{-3}{2} & 0 & 0 & \frac{-1}{2} & 1 \\
z_1 = & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 2 \\
\hline
\end{array}
$$

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Solution is $z = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

2 So $x^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $u^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Constraint $2x_1 - x_2 \geq 4$ is active at this solution. The solution doesn't change if we enforce it as an equality constraint.
Any point that's feasible for the modified problem would also be feasible for the original problem, and thus would have a higher objective than
4 the objective attained by x^*.

3. Write problem in standard form.

$$z(t) = \min \quad 3x_1 - 2x_2$$
$$\text{s.t.} \quad x_1 - x_2 \geq 7 + t$$
$$-x_1 \geq -5 + t.$$

(2)

Choose $t = -7$ first, to make rhs non-positive.

	x_1	x_2	1	t	$(t=-7)$
$x_3 =$	1	-1	-7	-1	0
$x_4 =$	-1	0	5	-1	12
$z =$	3	-2	0	0	

↑

	x_1	x_3	1	t	$(t=-7)$
$x_2 =$	1	-1	-7	-1	0
$x_4 =$	-1	0	5	-1	12
$z =$	1	2	14	2	

tableau is optimal for $-7-t \geq 0 \iff t \leq -7$

$5 - t \geq 0 \iff t \leq 5$.

So for $t \in (-\infty, -7)$, solution is $x = \begin{pmatrix} 0 \\ -7-t \end{pmatrix}$

(b)

objective $14 + 2t$.

As t increases through -7, first row of rhs becomes negative. Pivot on $(1,1)$.

	x_2	x_3	1	t
$x_1 =$	1	1	7	1
$x_4 =$	-1	-1	-2	-2
$z =$	1	3	21	3

Tableau is optimal provided $7+t \geq 0 \Rightarrow t \geq -7$
$-2-2t \geq 0 \Rightarrow t \leq -1$

So for $t \in (-7,-1)$ solution is

$$x = \begin{pmatrix} 7+t \\ 0 \end{pmatrix} \quad \text{objective } 21+3t.$$

(5)

As t increases through -1, second element on rhs becomes negative. No suitable pivot column, so problem becomes <u>infeasible</u>

(4)

Summary:

t	$x(t)$	$z(t)$
$(-\infty,-7]$	$\begin{pmatrix} 0 \\ -7-t \end{pmatrix}$	$14+2t$
$[-7,-1]$	$\begin{pmatrix} 7+t \\ 0 \end{pmatrix}$	$21+3t$
$(-1,\infty)$	infeasible.	

(3) $z(t)$ is convex, piecewise linear

4. Can express I as "the following LP has a solution":

$$\min 0'x \quad \text{s.t. } Ax \geq b$$
(P) $$x \geq l$$

dual of it is

(D) $$\max b'u + l'v$$

$$\text{st. } A'u + v = 0$$

$$u \geq 0, \; v \geq 0.$$

(6)

Case A: If I is true, then P has a solution with optimal objective 0. Thus, by strong duality (D) has a solution also, with optimal objective 0. Thus any vector (u,v) that is feasible for D must have $b'u + l'v \leq 0$, so II is not satisfied.

(7)

Case B: If I is false, then P is infeasible. By strong duality, then, D is either infeasible or unbounded. It is not infeasible, since $(u,v) = (0,0)$ is a feasible point. Hence it must be unbounded, so in particular there are vectors

(7)

(u,v) feasible for D that have a positive objective, in particular

$$b'u + l'v > 0, \quad A'u + v = 0, \quad u \geq 0, \; v \geq 0.$$