Final Examination

CS 525 - Spring 2009

Tuesday, May 12, 2009, 10:05am-12:05pm

Each question is worth the same number of points.

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, quoting any theorems you use.

1. Consider the following linear program:

$$\begin{array}{lrcl} \min & 2x_1 - x_2 & & \\ & x_1 - x_2 & \geq & 4, \\ \text{subject to} & 3x_1 + 2x_2 & \geq & 10, \\ & x_1, x_2 & \geq & 0. \end{array}$$

   (a) Write down the dual of this problem.

   (b) By using dual-labelled tableaus, find solutions for the primal and dual.

   (c) Suppose the right-hand side of the first constraint is changed from 4 to 10. Without performing any additional simplex iterations or referring to the tableau, give a lower bound on the optimal primal objective for the modified problem. Explain.

   (c) Does the solution of the primal problem change if you change the right-hand side of the second constraint to 12 (while leaving the right-hand side of the first constraint at 4)? Explain.

2. Solve

$$\begin{aligned} \min \ & \frac{1}{2}x_1^2 - 2x_1x_2 + 2x_2^2 + 4x_2 \\ \text{subject to} \ & x_1 + 2x_2 \leq 3, \\ & 2x_1 - x_2 \geq 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

   How does the solution change if the constraint $2x_1 - x_2 \geq 4$ is replaced by $2x_1 - x_2 = 4$?

3. Consider the following linear program with parametrized right-hand side:

$$\begin{aligned} z(t) = \min \ & 3x_1 - 2x_2 \\ \text{subject to} \ & x_1 - x_2 \geq 7 + t, \\ & x_1 \leq 5 - t, \\ & x_1 \geq 0, \ x_2 \geq 0. \end{aligned}$$

   (Note that the second constraint is a $\leq$ constraint.) Find $z(t)$ for all $t \in (-\infty, \infty)$. What properties does the function $z(t)$ have?

4. Consider the following two logical statements (where A is a given matrix and b and l are given vectors):

   I. There exists a vector x such that $Ax \geq b$, $x \geq l$;

   II. There exist vectors (u, v) such that $A'u + v = 0$, $u \geq 0$, $v \geq 0$, and $b'u + l'v > 0$.

   (Note the strict inequality in the last condition of II.) Show that *exactly one* of these two statements is true.